Combinatorial Games in Graphs: theory and ludic

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Semana PESC
October 2, 2018

Summary

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Academic trajectory

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What comes next?

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In January 2014, Professor Simone introduced me to Professor Sylvain.

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New game: Timber Game in caterpillars.

Academic trajectory

In January 2015: qualification.

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In March 2015: Grenoble.

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In Grenoble, in addition to working on Timber Game, we started working on Coloring Game.

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Working together with:

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Working together with:

- Clément Charpentier;
- Simon Schmidt.
- Math à modeler team.

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Back to Brasil in October 2015, we continued to work on combinatorial games, and write articles and my Ph.D. thesis.

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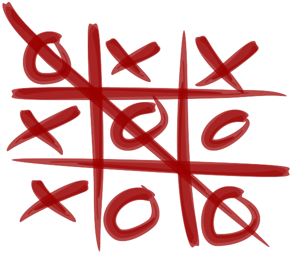

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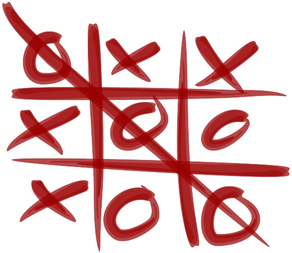

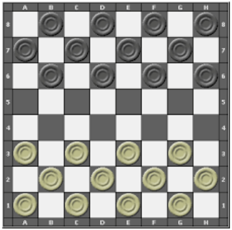

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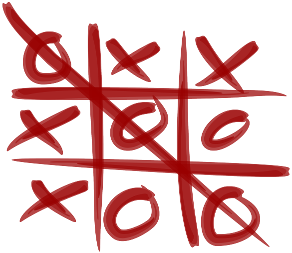

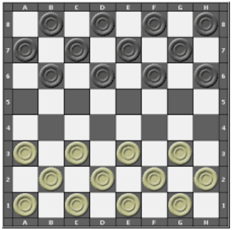

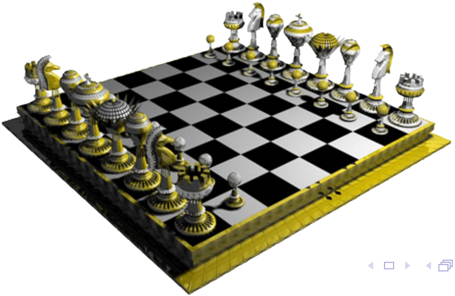

Goal

- Many researchers have been studying winning strategies in 2-player combinatorial games.
- We study the Timber Game and the Coloring Game in a caterpillar.
- Moreover, we study the Nordhaus-Gaddum type inequality to the parameter of these game.

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What is Coloring Game?

- The *coloring game* is a two player non-cooperative game conceived by Steven Brams.
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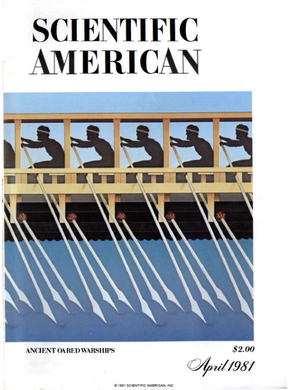

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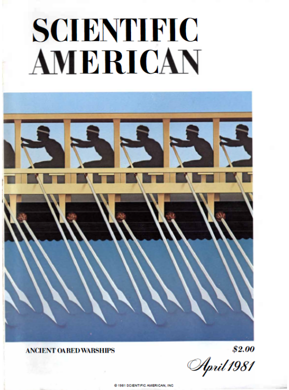

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How to play?

Given t colors, Alice and Bob take turns properly coloring an uncolored vertex.

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minimizer X maximizer

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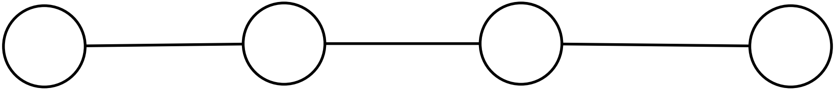

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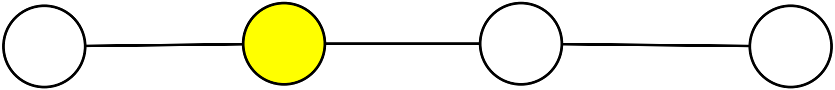

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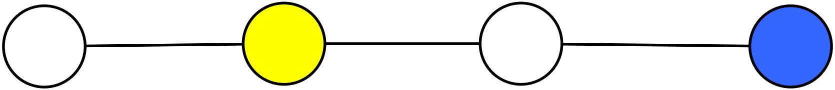

Who wins?

- Alice wins when the graph is completely colored with t colors; otherwise, Bob wins.
- The *game chromatic number* $\chi_g(G)$ of G is the smallest number t of colors that ensures that Alice wins (when Alice starts the game).

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$$\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$$

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Simple results

- $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$
  - $\chi_g(K_n) = n$
  - $\chi_g(S_n) = 1$
- $\chi_g(P_1) = 1$, $\chi_g(P_2) = \chi_g(P_3) = 2$
- For $n \geq 4$, we have that $\chi_g(P_n) = 3$
- $\chi_g(C_n) = 3$
- The stars $K_{1,p}$ with $p \geq 1$ are the only connected graphs satisfying $\chi_g(G) = 2$

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Different graph classes studied

- planar graphs: $7 \leq \chi_g(P) \leq 17$;
- outerplanar graphs: $6 \leq \chi_g(O) \leq 7$;
- toroidal grids: $\chi_g(TG) = 5$;
- partial k-trees: $\chi_g(P) \leq 3k + 2$;
- the cartesian products of some classes of graphs: for example, $\chi_g(T_1 \square T_2) \leq 12$;

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Literature for trees

- Bodlaender (1991): $\chi_g(T) \leq 5$.
- Faigle et al. (1993): $\chi_g(F) \leq 4$.
- Dunn et al.(2015): criteria for determining $\chi_g(F)$, for a forest **without vertex of degree 3**, in polynomial time.

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Our problem

- Due to the difficulty concerning this subject, the problem of characterizing forests with $\chi_g(F) = 3$ remains open.
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Caterpillar

- A *caterpillar* $cat(k_1, k_2, \ldots, k_s)$ is a tree which is obtained from a central path $v_1, v_2, v_3, \ldots, v_s$ (called spine), and by joining v_i to k_i new vertices, $i = 1, \ldots, s$.

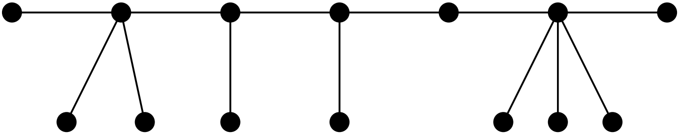

Figure: $cat(0, 2, 1, 1, 0, 3, 0)$.

Why caterpillars?

- Example presented in Bodlaender (1991) to prove the existence of a tree H_d with $\chi_g(H_d) \geq 4$:

- Dunn et al.(2015) proved that this caterpillar is the smallest tree such that $\chi_g(T) = 4$.

- We are interested in characterizing when $\chi_g(H)$ is 3 or 4.

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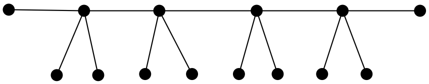

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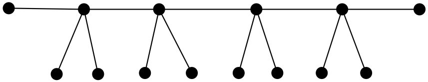

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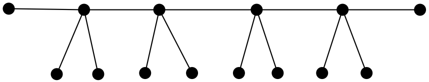

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What did we do?

- We have determined:
  - two sufficient conditions for $\chi_g(H) = 4$ for any caterpillar H;
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- Caterpillars

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Caterpillar with maximum degree 3

Theorem (Furtado et al., 2017)

Let H be the caterpillar $cat(k_1, ..., k_s)$ with $\Delta(H) = 3$. We have that H has $\chi_g^a(H)$, $\chi_g^b(H) \leq 3$. Moreover, let F be the forest where each connected component is a caterpillar and $\Delta(F) = 3$. We have that F has $\chi_g^a(F) \leq 3$.

Caterpillar without vertex of degree 2

Theorem (Furtado et al., 2017)

Let H be the caterpillar without vertex of degree 2. We have that $\chi_g^a(H) = \chi_g^b(H) = 4$ if, and only if, H is caterpillar $cat(k_1, \ldots, k_s)$, such that $k_1 = k_s = 0$, $k_i \neq 0$, $\forall i \in \{2, \ldots, s-1\}$, and there are at least four vertices of degree at least 4.

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Caterpillar without vertex of degree 3

Let *Family Q* be the set of caterpillars H_d, H_{33}, $H_{[\alpha]} \cup H_{[\beta]}$, $H_{[\alpha][\beta]}$ and $H_{[\alpha]3[\beta]}$.

Caterpillar without vertex of degree 3

Let *Family Q* be the set of caterpillars H_d, H_{33}, $H_{[\alpha]} \cup H_{[\beta]}$, $H_{[\alpha][\beta]}$ and $H_{[\alpha]3[\beta]}$.

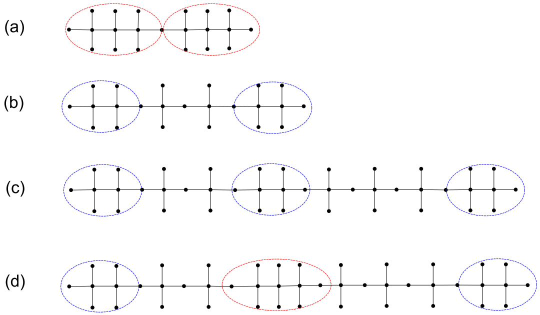

Figure: Caterpillars (a) H_{33} (b) $H_{[3]}$ (c)$H_{[3][4]}$ (d)$H_{[3]3[4]}$.

Caterpillar without vertex of degree 3

Theorem

A caterpillar H without vertex of degree 3 has $\chi_g(H) = 4$ *if, and only if, H has a caterpillar of Family Q as an induced subcaterpillar.*

Caterpillar with vertices of degree 1, 2, 3 and 4

Let *Family* Q' be the set of caterpillars $\{H'_{[\alpha]} \cup H'_{[\beta]},\ H'_{[\alpha]} \cup H_3,\ H_3 \cup H_3,$ H'_{22} and $H'_{[\alpha][\beta]}, H'_{23}\}$.

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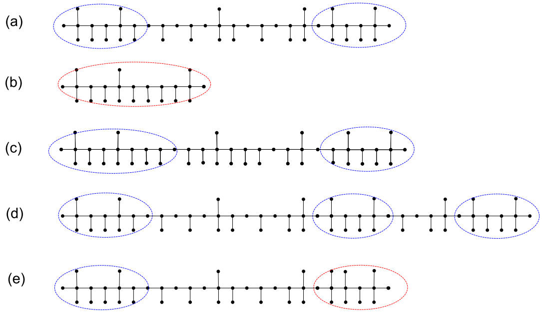

Figure: Caterpillars (a) $H'_{[6]}$ (b) H'_3 (c)H'_{22} (d)$H_{[6][3]}$ (e)H'_{23}.

Caterpillar with vertices of degree 1, 2, 3 and 4

Theorem (Furtado et al., 2017)

Let H be a caterpillar with vertices of 1, 2, 3 and 4. If H has a caterpillar of Family Q' as a induced subcaterpillar, then $\chi_g(H) = 4$.

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Summary

$\Delta(H)$	$\chi_g(H) = 1$	$\chi_g(H) = 2$	$\chi_g(H) = 3$	$\chi_g(H) = 4$
0	P_1	-	-	-
1	-	P_2	-	-
2	-	P_3	$P_n, n \geq 4$	-
3	-	star	not a star	-
4	-	star	see next Figure	see next Figure

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Summary

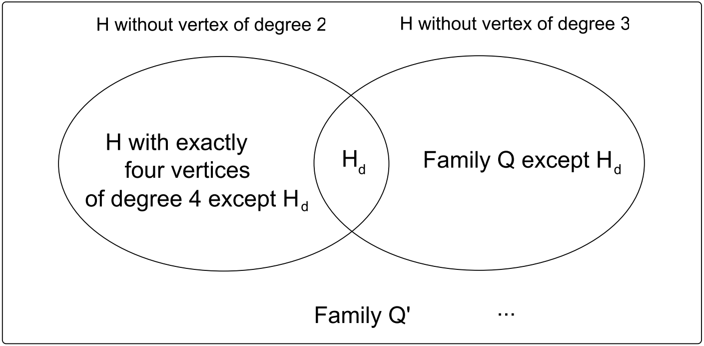

Figure: Caterpillars with $\Delta(H) = 4$ and $\chi_g(H) = 4$.

$\chi_g(F)$

Theorem (Furtado et al., 2017)

Let F be a forest composed by r trees T_1, ..., T_r. Assume that $\chi_g^a(T_1) \leq \chi_g^a(T_2) \leq ... \leq \chi_g^a(T_r)$, and, if there exist two trees with the same game chromatic number, then T_i and T_j are ordered in a way that $\chi_g^b(T_i) \leq \chi_g^b(T_j)$, for $i < j$. We have that:

1. *If $\chi_g^b(T_r) > \chi_g^a(T_r), \chi_g^b(T_{r-1})$, then $\chi_g(F) = \chi_g^a(T_r)$;*
2. *If $\chi_g^b(T_r) = \chi_g^b(T_{r-1}) > \chi_g^a(T_r)$, then $\chi_g(F) = \chi_g^b(T_r)$;*
3. *If $\chi_g^a(T_r) = \chi_g^b(T_r)$, then $\chi_g(F) = \chi_g^a(T_r) = \chi_g^b(T_r)$;*
4. *If $\chi_g^b(T_r) < \chi_g^a(T_r)$ and $\sum_{i=1}^{r-1} |V(T_i)|$ is even, then $\chi_g(F) = \chi_g^a(T_r)$;*
5. *If $\chi_g^b(T_r) < \chi_g^a(T_r)$ and $\sum_{i=1}^{r-1} |V(T_i)|$ is odd, then $\chi_g(F) = max \left\{\chi_g^a(F \backslash T_r), \chi_g^b(T_r)\right\}$.*

Summary

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What are Nordhaus-Gaddum type inequalities?

- Nordhaus and Gaddum (1956) showed lower and upper bounds on the sum of the chromatic number of a graph and its complement:

- Survey by Aouiche and Hansen (2013): 360 articles.
- To the best of our knowledge, the only Nordhaus-Gaddum type inequality existing for invariants related to games on graphs is by Alon et al.(2002) and concerns the game domination number.

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Theorem (Nordhaus and Gaddum, 1956)

If G is a graph of order n, then $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$. These bounds are best possible for infinitely many values of n.

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Nordhaus-Gaddum type inequalities to $\chi_g(G) + \chi_g(\overline{G})$

Theorem (Furtado et al., 2017)

For any graph G of order n, we have that $2\sqrt{n} \leq \chi_g(G) + \chi_g(\overline{G}) \leq \lceil \frac{3n}{2} \rceil$. *Moreover, the bounds are best possible asymptotically:*

1. *for infinitely many values of n, there are graphs G of order n with* $\chi_g(G) + \chi_g(\overline{G}) = \left\lceil \frac{4n}{3} \right\rceil - 1$;
2. *for infinitely many values of n, there are graphs G of order n with* $\chi_g(G) + \chi_g(\overline{G}) = 2\sqrt{2n} - 1$.

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The lower bound follows from Theorem of Nordhaus and Gaddum (1965) and the inequality $\chi(G) \leq \chi_g(G)$.

Nordhaus-Gaddum type inequalities to other games

- We determine the Nordhaus-Gaddum type inequalities to
  - the number of P-positions of a caterpillar (Timber Game);
  - the *game coloring number* of any graph G (Marking Game).
- *Marking Game* is a "colorblind" version of the coloring game.
- All bounds are tight, except the upper bound for the number of P-positions of a caterpillar.

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Why games?

Figure: Salon International de la Culture et des jeux mathématiques, Paris, 2015.

Figure: Festival da Matemática, Rio de Janeiro, 2017.

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THANK YOU!

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