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Note

Improved pebbling bounds

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Abstract

Consider a configuration of pebbles distributed on the vertices of a connected graph of order n. A pebbling step consists of removing two pebbles from a given vertex and placing one pebble on an adjacent vertex. A distribution of pebbles on a graph is called solvable if it is possible to place a pebble on any given vertex using a sequence of pebbling steps. The pebbling number of a graph, denoted $f(G)$, is the minimal number of pebbles such that every configuration of $f(G)$ pebbles on G is solvable. We derive several general upper bounds on the pebbling number, improving previous results.
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1. Introduction

1.1. Definitions

Given a connected graph $G = (V, E)$, let $D : V \to \mathbb{N}$ be a distribution of $\sum_v D(v)$ identical pebbles on the vertices of G. A *pebbling step* consists of removing two pebbles from a given vertex and placing one of these pebbles on an adjacent vertex (the other pebble is removed from the graph.) Given a root vertex $v \in V$, we say that D is *v-solvable* if we can place at least one pebble on v after some number of pebbling steps. D is called solvable if D is v-solvable for all $v \in V$. Let the *size* of a distribution D be $|D| = \sum_v D(v)$, the total number of pebbles on the graph. Then the pebbling number $f(G)$ is defined to be the smallest integer N such that any distribution of size N is solvable. Also, we define $f(G, v)$ to be the smallest integer N such that any distribution of size N is v-solvable. Throughout this paper, $G = (V, E)$ denotes a simple, connected graph, $n = |V|$ is the number of vertices in G, and $d = \mathrm{diam}(G)$ is the diameter of G.

1.2. Known bounds on the pebbling number

First we state an elementary bound on $f(G)$, given in [4]: clearly, if $D(v) = 0$ for the root vertex v and $D(w) = 1$ for all other vertices w, then D is unsolvable. Also, given vertices v_1 and v_2 at distance $d = \mathrm{diam}(G)$ from each other,

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if $D(v_1) = 2^d - 1$ and $D(w) = 0$ for every other vertex, then vertex v_2 cannot be pebbled and D is unsolvable. These facts were noted by Chung [2]. For an upper bound, note that if $|D| = (n-1)(2^d - 1) + 1$, then either each vertex has at least one pebble on it, or, by the pigeonhole principle, there exists a vertex with 2^d pebbles on it, and in either case D is solvable. To summarize,

$$\max\{n, 2^d\} \leqslant f(G) \leqslant (n-1)(2^d - 1) + 1. \tag{1}$$

Note that the upper bound in (1) is sharp if G is the complete graph K_n, but is way off target if $G = P_n$, the path on n vertices.

2. New upper bounds

In this section, we prove four new upper bounds on the pebbling number, two of which always improve the bound stated above, and with the rest doing better than (1) in most cases.

Theorem 1. $f(G) \leqslant (n - d)(2^d - 1) + 1$.

Proof. Chung [2] first noted that for the path P_n on n vertices, $f(P_n) = 2^{n-1}$. A simple proof of this result can be found in [4]. Now, given any root vertex $v \in V$, consider a set $S_v = \{Q_1, \ldots, Q_m\}$ of m paths on G, with the following two properties:

- each path has one endpoint at v; and
- each vertex in G is on at least one path.

These paths may overlap, although that is not necessarily the case. Let q_i be the length of the path Q_i. If some path Q_i has 2^{q_i} pebbles on it, then v can be reached using that path. Thus, the pigeonhole principle guarantees that

$$f(G, v) \leqslant \left(\sum_{i=1}^{m} (2^{q_i} - 1) \right) + 1.$$

To construct one such path set, let Q_1 be the path from v to some vertex w at maximum distance from v, that is, $\mathrm{dist}(v, w) = e(v)$ where $e(v) = \max\{\mathrm{dist}(v, w)\}$ is the eccentricity of v. Then there are $n - e(v) - 1$ vertices not on that path. In the worst case, i.e., in the sense of maximizing the number of paths, each of those vertices requires a distinct path of length no longer than $e(v)$ to connect it to v. Thus, we have in total $n - e(v)$ paths of length at most $e(v)$, and

$$f(G, v) \leqslant \left(\sum_{i=1}^{n-e(v)} (2^{e(v)} - 1) \right) + 1$$
$$= (n - e(v))(2^{e(v)} - 1) + 1.$$

It follows that

$$\begin{aligned} f(G) &= \max\{f(G, v)\} \\ &\leqslant \max\{(n - e(v))(2^{e(v)} - 1) + 1\} \\ &= (n - d)(2^d - 1) + 1, \end{aligned}$$

where the last equality above follows from the fact that the function $(n - j)(2^j - 1)$ is monotone increasing for *integers j* with $1 \leqslant j \leqslant n - 1$; to see this, we set $\phi(j) = (n - j)(2^j - 1)$ and note that $\phi(j + 1) \geqslant \phi(j)$ if and only if $(n - j)2^j \geqslant 2^{j+1} - 1$, and thus if $n - j \geqslant 2$, or $j \leqslant n - 2$.

Note that the upper bound of Theorem 1 obviously improves (1), and is sharp for both $G = K_n$ and $G = P_n$. $\square$

Theorem 2.

$$f(G) \leqslant \left(n + \left\lfloor \frac{n-1}{d} \right\rfloor - 1 \right) (2^{d-1}) - n + 2.$$

Proof. Given a root vertex $v \in V$ and $k \geqslant 1$, let $\{w_1, \ldots, w_k\} \subset V$ be a set of vertices such that $\operatorname{dist}(w_i, v) = e(v)$ and there exists a set $\{p_1, \ldots, p_k\}$ of length-$e(v)$ paths, with path p_i connecting w_i and v, such that no two paths share any vertex except v. Such a set of vertices must exist for, in the worst case we may have $k = 1$. Then the number of such paths, k, must satisfy $k \leqslant c$, where $c = \lfloor (n-1)/e(v) \rfloor$.

Now, we claim that

$$f(G, v) \leqslant (n-1)(2^{e(v)-1} - 1) + c(2^{e(v)-1}) + 1.$$

To see this, observe that if v already has a pebble on it or if any vertex has $2^{e(v)}$ pebbles on it, then we know that D is v-solvable. If neither is the case, then (with $|D| = (n-1)(2^{e(v)-1} - 1) + c(2^{e(v)-1}) + 1$), the pigeonhole principle guarantees that there exists a set of $c+1$ vertices with at least $2^{e(v)-1}$ pebbles each. This is because we get a contradiction if v has no pebbles, there are at most c vertices with at most $2^{e(v)} - 1$ pebbles each, and the remaining vertices have at most $2^{e(v)-1} - 1$ pebbles each.

If any of these $c + 1$ vertices is at distance at most $e(v) - 1$ from v, we are done. If all $c + 1$ vertices are at distance $e(v)$ from v, then there must exist vertices u_1 and u_2, each with at least $2^{e(v)-1}$ pebbles, with length-$e(v)$ paths p_{u_1} and p_{u_2} to v that both pass through some vertex w adjacent to v. Then we can place 2 pebbles on w, and hence we can place 1 pebble on v. Thus, D is v-solvable and we have

$$
\begin{aligned}
f(G) &= \max\{f(G, v)\} \\
&\leqslant \max\left\{\left(n + \left\lfloor \frac{n-1}{e(v)} \right\rfloor - 1\right)(2^{e(v)-1}) - n + 2\right\} \\
&= \left(n + \left\lfloor \frac{n-1}{d} \right\rfloor - 1\right)(2^{d-1}) - n + 2,
\end{aligned}
$$

as asserted. $\square$

Note that Theorem 2 is sharp if $G = K_n$, but *not* if $G = P_n$; in general, it is easy to verify that Theorem 2 performs better than Theorem 1 whenever d is small and n is large enough. Also, it is easy to check that Theorem 2 always improves the upper bound given by (1): to see this, we simply rewrite the latter as $(2n - 2)2^{d-1} - n + 2$ and note that

$$(2n - 2)2^{d-1} - n + 2 \geqslant \left(n + \left\lfloor \frac{n-1}{d} \right\rfloor - 1\right)(2^{d-1}) - n + 2,$$

with equality holding if and only if $d = 1$.

A set $S \subset V$ is said to be a *dominating set* if each vertex $v \in V$ is a member of S or is adjacent to at least one member of S. S is said to be a *perfect dominating set* if each vertex $v \in V$ is a member of S or is adjacent to *exactly* one member of S. An *independent dominating set* is a dominating set whose vertices are independent. In Theorem 3, we will need the set S to be independent as well as perfect; in other words, we must have an *efficient dominating set* (see [3] for more on perfect, independent, and efficient domination.) Both Theorems 3 and 4 will make use of a result of Moews [5]: the k-pebbling number $f_k(G)$ of a graph G, first defined in [2], is the minimum number of pebbles that must be placed on G so that any vertex can have k pebbles placed on it in a series of pebbling moves, regardless of the initial configuration of pebbles. Moews' result on the k-pebbling number of trees reduces in the special case of the star $K_{1,r}$ to

$$f_k(K_{1,m-1}) = 4k + m - 3. \tag{2}$$

Theorem 3. *Suppose G has an efficient dominating set of size γ. Then*

$$f(G) \leqslant 2^{d+1}\gamma + n - 4\gamma + 1.$$

Proof. Let $S = \{s_1, \ldots, s_\gamma\}$ be an efficient dominating set. For each s_i, let $A_i = \{s_i\} \cup \{w : w$ is adjacent to $s_i\}$. The A_is partition the vertex set V, with each A_i having diameter at most 2. Also, each A_i contains $K_{1,|A_i|-1}$ as a subgraph. Thus by (2), if there exists a set A_i containing at least $2^{d+1} + |A_i| - 3$ pebbles in total, then we may place 2^{d-1} pebbles on any vertex in A_i. Furthermore, for $v \notin A_i$, v is at distance at most d from s_i, so v is at distance at most $d - 1$ from some

vertex in A_i—since S is an efficient dominating set. Therefore, we may place a pebble on any $v \in V$. The pigeonhole principle now guarantees that

$$f(G) \leqslant \left(\sum_{i=1}^{\gamma} (2^{d+1} + |A_i| - 4) \right) + 1$$
$$= 2^{d+1}\gamma + n - 4\gamma + 1,$$

as required.   □

Note that Theorem 3 yields a better bound than that given by (1) if and only if

$$\gamma \leqslant \frac{(2^d - 2)n - (2^d - 1)}{2^{d+1} - 4} = \frac{n-1}{2} + \frac{1}{2^{d+1} - 4}. \tag{3}$$

Now (3) fails to hold on a non-trivial connected graph only if $d = 1$, when the right side is meaningless, or $d \geqslant 2$ and $\gamma \geqslant n/2$, i.e., for graphs where every vertex in the dominating set has exactly one neighbor. In the former case, (1) *does* do better than Theorem 3—but only because the proof of Theorem 3 only used the star structure of the sets A_i, and did not exploit the fact that there might be edges between the vertices w adjacent to s_i, thus causing the diameter of A_i to conceivably equal one. Consider the case where $\gamma = n/2$. Since the only connected graph with an efficient dominating set and domination number $n/2$ is the corona $G \circ K_1$ (see pp. 41–42 of [3] for a further discussion), we see that Theorem 3 does better than (1) "almost all" the time.

Our final result shows how Theorem 3 may be generalized, at little cost, to graphs that do not admit a perfect independent dominating set.

Theorem 4. *Suppose G has a dominating set of size γ. Then*

$$f(G) \leqslant 2^{d+1}\gamma + n - 3\gamma + 1.$$

Proof. Let $S = \{s_1, \ldots, s_\gamma\}$ be a dominating set. For a fixed vertex v, pick paths $P_1, P_2, \ldots, P_\gamma$ (of length at most d) of the form $P_i = v a_1 a_2 \ldots a_{L_i} w_i s_i$. (In case v is a neighbor of s_i or is s_i itself, the path reduces to $v s_i$ or s_i, and w_i is defined to be s_i in either case. If $\mathrm{dist}(v, s_i) = 2$, there are no a_js in the path.) Now define

$$A_1 = N_{s_1} \cup \{w_1, s_1\};$$

$$A_2 = (N_{s_2} \setminus N_{s_1}) \cup \{w_2, s_2\};$$

$$A_3 = (N_{s_3} \setminus (N_{s_1} \cup N_{s_2})) \cup \{w_3, s_3\};$$

$$\vdots$$

$$A_i = \left(N_{s_1} \setminus \bigcup_{j=1}^{i-1} N_{s_j} \right) \cup \{w_i, s_i\};$$

$$\vdots$$

$$A_\gamma = \left(N_{s_\gamma} \setminus \bigcup_{j=1}^{\gamma-1} N_{s_j} \right) \cup \{w_\gamma, s_\gamma\},$$

where N_x consists of all neighbors of x that do not belong to the set S. Notice that the A_is are again sets of diameter at most 2. The rest of the argument follows the proof of Theorem 3 very closely: if there exists a set of vertices A_i containing at least $2^{d+1} + |A_i| - 3$ pebbles in total, then we may place 2^{d-1} pebbles on any vertex in A_i. Now v

is at distance at most $d - 1$ from at least one vertex in A_i, namely w_i. Therefore, we may place a pebble on v. The pigeonhole principle now guarantees that

$$f(G, v) \leqslant \left(\sum_{i=1}^{\gamma} (2^{d+1} + |A_i| - 4) \right) + 1$$

$$= \gamma 2^{d+1} - 4\gamma + 1 + \sum_{i=1}^{\gamma} |A_i|$$

$$\leqslant \gamma 2^{d+1} - 4\gamma + 1 + \sum_{i=1}^{\gamma} \left(\left| \left(N_{s_i} \setminus \bigcup_{j=0}^{i-1} N_{s_j} \right) \cup \{s_i\} \right| + 1 \right)$$

$$= \gamma 2^{d+1} - 3\gamma + 1 + n,$$

and thus that $f(G) \leqslant 2^{d+1}\gamma + n - 3\gamma + 1$, as claimed. $\quad\square$

As with Theorem 3, Theorem 4 might not always yield a bound better than that given by (1). It is easy to verify, however, that this occurs whenever

$$\gamma \leqslant \frac{(2^d - 2)n - (2^d - 1)}{2^{d+1} - 3}. \tag{4}$$

Even for $d = 2$, (4) holds if $\gamma \leqslant (2n - 3)/5$, which is encouraging since $\gamma \leqslant 2n/5$ for "most" connected graphs with minimum degree at least 2 (see pp. 41–42 of [3]). As d increases, moreover, we see that (4) holds unless γ is close to $n/2$. Thus, Theorem 4 improves on (1) in many cases.

In a similar fashion, Theorems 3 and 4 may, respectively, be checked to outperform Theorem 1 if

$$\gamma \leqslant \frac{(2^d - 2)n - d(2^d - 1)}{2^{d+1} - 4}$$

and

$$\gamma \leqslant \frac{(2^d - 2)n - d(2^d - 1)}{2^{d+1} - 3},$$

while Theorems 3 and 4 do better than Theorem 2 if, respectively,

$$\gamma \leqslant \frac{(2^{d-1} - 2)n - (\lfloor (n - 1)/d \rfloor - 1) 2^{d-1} + 1}{2^{d+1} - 4}$$

and

$$\gamma \leqslant \frac{(2^{d-1} - 2)n - (\lfloor (n - 1)/d \rfloor - 1) 2^{d-1} + 1}{2^{d+1} - 3}.$$

Theorems 3 and 4 thus often do better than Theorems 1 and 2, but are they ever tight? This question was raised by one of the referees. We provide a partial answer. For the complete graph, our baseline test case, we have $f(K_r) = r$ but using Theorem 3 with $\gamma = 1$ and $d = 1$ yields a bound of $r + 2^{d+1}\gamma - 4\gamma + 1 = r + 1$. The next obvious case to check is $K_{1,r}$, whose pebbling number equals $r + 2$. With $d = 2$ and $\gamma = 1$, however, we see that the upper bound of Theorem 3 is $r + 6$, so we only have asymptotic tightness. The same asymptotic tightness holds if we consider the graph G consisting, for even r, of two $K_{1,r/2}$'s connected at their roots by an edge $v_1 - v_2$; it is easy to verify that $f(G) = r + 6$, with the worst case configuration being no pebbles on v_1 or v_2, one pebble at each of $(r/2) - 1$ vertices on the $K_{1,r/2}$'s and 8 pebbles on any other vertex. However Theorem 4 applied with $\gamma = 2$ yields a bound of $r + 29$ for this diameter 3 graph on $r + 2$ vertices. In general we believe that the nature of the proof of either theorem, which uses the pigeonhole principle in a worst case scenario fashion, is unlikely to result in a tight result. Possible directions for improvement are suggested below.

3. Open problems

It would be interesting to develop general bounds on the pebbling number that are not in terms of the diameter of the graph. We have made some progress in this matter by proving a hybrid bound that depends also on the domination number, but much more needs to be done in this regard. Of particular interest would be bounds that depend on the girth of the graph, or are expressed in terms of more robust graph invariants such as the tree width (see Robin Thomas' NSF-CBMS lecture notes at http://www.math.gatech.edu/~thomas/SLIDE/CBMS/ (a book [7] is forthcoming) for an exposition of tree decompositions and tree width).

Also, how much of an improvement can be made in Theorems 3 and 4 by considering decompositions into sets of (even) diameter four or higher, rather than into the diameter two sets A_i considered in the proofs of these theorems? This would be necessary if one considers distance k-domination, $k \geqslant 2$. Progress along these lines might be contingent on obtaining tight upper bounds, analogous to those obtained in [6] for $d = 2$, on the pebbling numbers of graphs with diameter four or higher (the diameter three case has been solved recently by Bukh [1]). Or perhaps we might be able to use the embedded tree structure of the k-domination graph of a vertex v, together with results in [5] along the lines of (2), to make the needed improvements.

Acknowledgments

References

[1] B. Bukh, Maximum pebbling number of graphs of diameter three, J. Graph Theory 52 (2006) 353–357.
[2] F. Chung, Pebbling in hypercubes, SIAM J. Discrete Math. 2 (1989) 467–472.
[3] T. Haynes, S. Hedetniemi, P. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
[4] G. Hurlbert, A survey of graph pebbling, Congr. Numer. 139 (1999) 41–64.
[5] D. Moews, Pebbling graphs, J. Combin. Theory (B) 55 (1992) 244–252.
[6] L. Pachter, H.S. Snevily, B. Voxman, On pebbling graphs, Congr. Numer. 107 (1995) 65–80.
[7] R. Thomas, Graph structure and decomposition, forthcoming text.