ON PEBBLING GRAPHS

by

Lior Pachter
Department of Mathematics
MIT
Cambridge, MA 02139-4307

and

Hunter S. Snevily and Bill Voxman
Department of Mathematics and Statistics
University of Idaho
Moscow, ID 83844
email: snevily@uidaho.edu

The *pebbling number* of a graph G, $f(G)$, is the least m such that, however m pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. We give another proof that $f(Q^n) = 2^n$ (Chung) and show that for most graphs $f(G) = |V(G)|$ or $|V(G)|+1$. We also find $f(G)$ explicitly for certain classes of graphs (i.e. for odd cycles and squares of paths), characterize efficient graphs, show that most graphs have the 2-pebbling property, and obtain some results on optimal pebbling.

I. Introduction

Throughout this paper, unless stated otherwise, G will denote a simple connected graph on n vertices and $f(G)$ will denote the pebbling number of G (defined below).

Suppose p pebbles are distributed onto the vertices of a graph G. A *pebbling move* (step) consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. We say a pebble can be moved to a vertex v, the target vertex, if we can apply pebbling moves repeatedly (if necessary) so that in the resulting configuration the vertex v has one pebble. In this paper, the letter v will frequently be used to denote the target vertex of the graph under consideration, context should make it clear. For a graph G, we define the *pebbling number* $f(G)$ to

CONGRESSUS NUMERANTIUM **107**(1995), pp.65-80

be the smallest integer m such that for any distribution of m pebbles to the vertices of G, one pebble can be moved to any specified vertex v.

The first paper on pebbling [1] was motivated by the following question of Lagarias and Saks: Is $f(Q^n) = 2^n$? (Here Q^n denotes the n-dimensional hypercube.) Chung answered this question in the affirmative, by proving a stronger result. Before we state Chung's result we need two definitions. For any two graphs G_1 and G_2, we define the *cartesian product* $G_1 \times G_2$ to be the graph with vertex set $\{(v_1, v_2) : v_1 \in V(G_1),\ v_2 \in V(G_2)\}$ and there is an edge between (v_1, v_2) and (v_1^1, v_2^1) if and only if $(v_1 = v_1^1$ and $(v_2, v_2^1) \in E(G_2)) or (\{v_1, v_1^1\} \in E(G_1)$ and $v_2 = v_2^1)$. It is easy to see that the 1–cube is K_2; the 2–cube is $K_2 \times K_2$; and the n–cube Q^n is $Q^{n-1} \times K_2$.

We say a graph G satisfies the *2-pebbling property* if two pebbles can be moved to a specified vertex when the total starting number of pebbles is $2f(G) - q + 1$, where q is the number of vertices with at least one pebble. Chung proved the following results.

Theorem 1. (Chung [1]) *Suppose G satisfies the 2–pebbling property. Then the following holds: (i) $f(G \times K_t) \le tf(G)$*
(ii) If $f(G \times K_t) = tf(G)$, then $G \times K_t$ satisfies the 2-pebbling property.

Corollary 1. $f(Q^n) = 2^n$.

Chung used her theorem to give a novel proof of a number theoretic result of Lemke and Kleitman [3]. See [1] for details.

In section 2 below we give a very simple proof of Corollary 1. In section 3, we show that most graphs have the 2-pebbling property and that for most graphs $f(G) = n$ or $n+1$. We also show that if $|E(G)| \ge \binom{n-1}{2} + 2$ then $f(G) = n$. In section 4 we discuss efficient graphs, this involves a variant of the original pebbling problem; here we assign fractional weight to the pebbles. In section 5 we find $f(G)$ explicitly for certain types of graphs, for example, when G is an odd cycle.

In section 6 we introduce the notion of an optimal pebbling number $of(G)$ of a graph. In this case, one is allowed to choose where the pebbles may be distributed as long as a pebble can be moved to any desired vertex. We conclude our paper with several conjectures and open problems.

2. $f(Q^n) = 2^n$

The n–cube Q^n can be considered as the graph whose vertices are labeled by the binary n–tuples and such that two vertices are adjacent if and only if their corresponding n–tuples differ in precisely one position. We may also designate the vertices of Q^n by subsets of $[n] = \{1, 2, \ldots, n\}$. In our proof below we will use this second interpretation. A collection I, of subsets of a finite set X, is called an ideal (hereditary family) if $A \in I,\ B \subset A \Rightarrow B \in I$. Thus ideals can be interpreted as induced subgraphs of Q^n. Keeping this in mind we can now prove our first theorem.

Theorem 2. *Let I be an ideal of an n–element set with $|I| = p$.*

(i) No matter how p pebbles are placed at the vertices of I one pebble can always be moved to ∅ (the empty set).

(ii) Suppose we have placed $p + h + 1$ pebbles at the vertices of I such that h vertices contain no pebbles at all. Then two pebbles can be moved to ∅.

Proof. Let k denote the maximum size of an element in I. We will proceed by induction on $|I|$. Clearly, our claims are true if $n = 1$ or if $k = 0$ or 1 (for any n). Let I be a minimal counterexample. Let S be those subsets of I that do not contain n and let T be those subsets of I containing n. Clearly S is an ideal with $|S| < |I|$ and T is isomorphic to an ideal (with $\{n\} \equiv \emptyset$) and $|T| < |I|$. Let $|S| = s$ and $|T| = t$, so $s + t = p$.

Proof of (i). Note that each vertex in T is adjacent to at least one vertex in S. If s pebbles are placed on the vertices of S we are done by induction. So assume that $s - r$ ($r \geq 1$) pebbles are placed on the vertices of S hence $t + r$ pebbles are placed on the vertices of T. By (ii) we see that T must contain at least r "empty" vertices (otherwise we could move two pebbles to $\{n\}$ and then one pebble to $\emptyset$).

Let U be the set of vertices in T that contain 2 or more pebbles each. The total number of pebbles at the vertices of U is at least $|U| + 2r$. It is easy to see that at least r pebbles can be moved from U to S and we are done.

Since I satisfies (i) it must fail at (ii). Suppose we have placed $p+h+1$ pebbles at the vertices of I such that h vertices contain no pebbles at all. Let S and T be defined as before. Let h_S be the number of vacant vertices in S and let h_T be the number of vacant vertices in T, so $h = h_S + h_T$.

First, suppose that $s-r$ ($r \geq 1$) pebbles are assigned to the vertices of S. Then $t+h+1+r$ pebbles are assigned to the vertices of T. Now $h = h_S + h_T$ and $r \leq h_S$, so T has $t + h_T + 1 + (r + h_S)$ pebbles and h_T vacant vertices. Let $A_i 1 \leq i \leq t$ be the elements in I that contain n and let $T_0 = T = \{A_1, A_2, \ldots, A_t = \{n\}\}$ and let $T_i = T_{i-1} - \{A_i\}$ for $1 \leq i \leq t-1$. We may assume that the A_i have been labeled so that each T_i is isomorphic to an ideal. (We think of each T_i as an ideal (induced subgraph) of T_{i-1}.) For all i, we have $\{n\} \in V(T_i)$ and $\{n\} \equiv \emptyset$. Let $q_i = 2m_i + r_i$ ($0 \leq r_i \leq 1$) be the number of pebbles at A_i. Let j be the least integer such that $\sum_{i=1}^{j} m_i \geq r$ (recall that T has at least $t + 2r$ pebbles). First, suppose $\sum_{i=1}^{j} m_i = r$. Then r pebbles can be moved from $A_1, A_2, \ldots A_j$ to S, leaving $|T_j| + h_{T_j} + 1$ pebbles at the vertices of T_j (here h_{T_j} is the number of vacant vertices of T_j) and s pebbles at S. By induction we are done. A similar argument can be given if $\sum_{i=1}^{j} mi > r$ (here we work with T_{j-1} instead of T_j) and we are done.

Therefore, we may assume that S contains $s+r$ ($r \geq 0$) pebbles and h_S empty vertices. Then by induction $r \leq h_S$, so T has at least $t + h_T + 1$ pebbles. By induction, we can move one pebble to $\emptyset$ in S and we can move two pebbles to $\{n\}$ in T (hence one additional pebble to $\emptyset$) and we are done.

Corollary 1. $f(Q^n) = 2^n$.

3. The 2–pebbling property and f(G) = n or n + 1.

We will show that having diameter 2 is a sufficient, but not necessary condition for a graph to satisfy the 2-pebbling property. We will make use of the following simple facts.

Fact 1. $f(G) \geq max\{2^{diam(G)}, n\}$

Proof. Obvious.

Fact 2. If G contains a cut vertex then $f(G) \geq n + 1$.

Proof. Let u be a cut vertex of G and let C_1 and C_2 be two distinct components of $G - u$. Let $x \in V(C_1)$ and let $v \in V(C_2)$ (v is our target vertex). Place one pebble at each vertex of $V(G) - \{u, x, v\}$ and place 3 pebbles at x. Clearly no pebble can be moved to v. □

Theorem 3. *Let G be a graph with diameter $(G) = 2$. Then G has the 2-pebbling property.*

Proof. Throughout this proof we will assume that v is our targent vertex. It is easy to verify that our theorem is true if $|V(G)| = n \leq 4$. Therefore we will assume that $|V(G)| = n \geq 5$. Let q be the number of vertices with at least one pebble. If v has a pebble on it we are done ($q \leq n \leq f(G)$ so $2f(G) - q \geq f(G)$). Therefore we may assume that v contains no pebbles and hence that $q \leq n - 1$. Next, suppose that a neighbor of v, say u, contained 2 or more pebbles. Then we see that we can move a pebble from u to v leaving us with $2f(G) - q + 1 - 2 \geq f(G)$ pebbles that haven't been moved and we are done. Therefore, we may assume that every neighbor of v contains at most one pebble.

Case (i): G has a cut-vertex.

Assume u is a cut-vertex of G. Since G has diamter 2, u has degree $n - 1$. If $u = v$ our proof is trivial ($\deg(u) = n - 1$ and ($f(G) \geq n + 1$). Thus, we may assume that $u \neq v$. Suppose u contains a pebble. Then since $\deg(u) = n - 1$ we can move a pebble to u and then move one pebble to v. This leaves us with $2f(G) - q + 1 - 3 \geq f(G)$ pebbles in G that have not been moved and we are done. Therefore we may assume that there are no pebbles on u, hence $q \leq n - 2$. Thus there are at least $q + 7$ pebbles on G. No matter how we try to distribute the seven "extra" pebbles onto the q "occupied" vertices we see that at least four pebbles can be moved to u, hence at least 2 pebbles can be moved to v and we are done.

Case (ii): G is 2–connected.

First assume that $q = n - 1$ (hence every neighbor of v has exactly one pebble). Now there are at least $q + 3$ pebbles on G. Suppose that some vertex, say w, contained four pebbles. Then since G is 2–connected we see that we can move one pebble from w to some neighbor of v, say x_1, and another pebble from w to another neighbor of v, say x_2 ($x_2 \neq x_1$). This gives us two neighbors of v each containing two pebbles and we are done. Therefore, we may assume that G contains two vertices, say w_1 and w_2, such that each one contains at least two pebbles. Using the 2–connectedness of G again, we see that we can move pebbles so that v has two neighbors each one of which contains at least two pebbles and we are done.

Therefore, we may assume that $q \leq n-2$. Let $x_1, x_2, \ldots, x_i$ be the neighbors of v. After relabeling, we may assume that the vertices $x_1, x_2, \ldots, x_j$ each contain a pebble and that the remaining neighbors of v do not. Assume $j = i$ (i.e. all neighbors of v contain a pebble). We know that G contains at least $q+5$ pebbles, hence at least three more pebbles can be moved to the neighbors of v (diameter$(G) = 2$) and we are done. Whence, we may assume that $q \leq n-2$ and that $j < i$.

Suppose $q = n-2$, then by our above remarks we know that v and x_i (a neighbor of v) are the only vertices without pebbles. Let w be a vertex containing two or more pebbles and suppose w is adjacent to x_j with $j < i$. Then we can move a pebble from w to x_j and then to v leaving $2f(G)-q+1-3 = 2f(G)-q-2 = 2f(G)-n \geq f(G)$ pebbles on G that have not been moved and we are done. Hence, if $q = n-2$, then all vertices containing two or more pebbles must be adjacent to x_i (the one neighbor of v that doesn't contain any pebbles). Remember that G contains at least $q+5$ pebbles ($q = n-2$).
There are 3 cases to consider.

Subcase (a). Some vertex of G, say w, contains 6 or more pebbles.

Proof. Since G is 2–connected, G must contain a path of the form $w-y-x_j$ ($j < i$), where y and x_j each contain one pebble. Thus we can move one pebble from w to x_j, leaving four pebbles at w and two pebbles at x_j. Now move two pebbles to x_i from w. Thus v has two neighbors each containing two pebbles and we are done. □

Subcase (b). Some vertex of G, say w, contains four or more pebbles and another vertex, say z, contains at least two pebbles.

Proof. G must contain a path of the form $z-y-x_j$ ($j < i$), where y and x_j each contain one pebble. Thus we can move a pebble from z to x_j, and two pebbles from w to x_i, hence we are done. □

Subcase (c). Three vertices of G, say w_1, w_2, w_3, each contain two or more pebbles.

Proof. G must contain a path of the form w_1-y-x_j ($j < i$) where y and x_j each contain one pebble. Move one pebble from w_1 to x_j and two pebbles to x_i (one from w_2 and one from w_3). Again, it is easy to see we are done. □

Therefore, we may assume that $q \neq n-2$ hence $q \leq n-3$. This means that G contains at least $2f(G)-q+1 \geq q+7$ pebbles and also that $2f(G)-q+1-4 \geq f(G)$.

As before, we see that if x_j is a neighbor of v containing a pebble then x_j is not adjacent to a vertex containing two or more pebbles. Now let u be a neighbor of v without any pebbles. If u is adjacent to a vertex containing four pebbles or two vertices (each containing at least two pebbles), then we can move a pebble to v (through u) at a "cost" of 4 pebbles. This leaves $2f(G)-q+1-4 \geq f(G)$ pebbles on G that have not been moved and we are done. Whence, we conclude that G cannot have any vertices with more than 3 pebbles on it and furthermore, if u is a neighbor of v without any pebbles, then u is adjacent to at most one vertex containing 2 or more pebbles. Therefore, we can assume that the number of vertices with 2 or more pebbles (but at most 3 pebbles) is less than or equal to the number of neighbors of v that contain no pebbles at all. Now suppose v has k ($k \geq 1$) neighbors that fail to have any pebbles. Then $q \leq n-(k+1)$ and $2f(G)-q+1 \geq q+2(k+1)+1$. Hence at least $k+2$ vertices in G contain 2 or more pebbles (but at most 3 pebbles) and this contradicts our previous statement. ∎

Next, we will show that if $\text{diam}(G)=2$ then $f(G)=n$ or $n+1$.

Lemma 1. *Let G be a graph with diam(G)= 2 and $|V(G)|=n \geq 6$. If we pebble G with the additional requirement that at least three vertices receive two or more pebbles each, then only n pebbles are needed so that a pebble may be moved to any desired vertex v.*

Proof. Assume that n pebbles are distributed on the vertices of G so that at least three vertices have two or more pebbles. If G has a cut-vertex our proof is trivial. Therefore assume G is 2–connected and that v is our target vertex. Clearly, no neighbor of v may contain 2 or more pebbles.

Let $x_1, x_2, \ldots, x_k$ ($k \geq 3$) be the vertices in our pebbling of G that contain 2 or more pebbles. Then each x_i must be adjacent to a neighbor of v, say y_i. Clearly, $y_i \neq y_j$ ($i \neq j$) and the y_i's are without pebbles. Since v is without pebbles we see that some x_j must contain 3 pebbles. (No x_i may contain four pebbles since $\text{diam}(G)=2$.) After relabeling, if necessary, we may assume that x_1 has 3 pebbles. Now x_1 cannot be adjacent to any other x_i, therefore, there must exist $k-1$ distinct vertices $z_{12}, z_{13}, \ldots, z_{1k}$ that are not neighbors of v and such that z_{1j} is adjacent to both x_1 and x_j. Clearly, the z_{1j}'s are without pebbles.

Thus x_j ($j \geq 2$) is adjacent to at least two vertices (y_j and z_{1j}) that fail to have any pebbles. Hence we may assume that each x_i ($1 \leq i \leq k$) has exactly three pebbles. Therefore none of the x_i's may be adjacent. Thus there must exist a vertex z_{23} distinct from $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k, z_{12}, z_{13}, \ldots, z_{1k}$ and v, that is adjacent to both x_2 and x_3. Clearly, z_{23} is without pebbles. But this leaves us with one more pebble to add to some x_i ($1 \leq i \leq k$) and we are done. ∎

Lemma 1 will be used to prove our next theorem.

Theorem 4. *Let G be a graph with diam(G)= 2, then $f(G)=n$ or $n+1$.*

Proof. Assume that v is our target vertex and that $n+1$ pebbles have been distributed on the vertices of G. Our proof is trivial if G contains a cut vertex. Therefore assume G is 2–connected and that $|V(G)|=n \geq 6$. (The cases $n=2,3,4$ and 5 can easily be checked.) Clearly, v and at least one neighbor of v, say u, fail to have any pebbles. By our previous lemma we may assume that at most two vertices in G have 2 or more pebbles. Let x_1, x_2 denote the vertices in G containing 2 or 3 pebbles. We may assume that x_1 has exactly three pebbles and x_2 has at least two pebbles. Now x_1 and x_2 are adjacent to distinct neighbors of v, say y_1 and y_2 (note that y_1 and y_2 are free of pebbles). Furthermore, there must exist a vertex $z_{12} \neq y_1, y_2$ that is adjacent to both x_1 and x_2. Note that z_{12} cannot have a pebble on it, else a pebble can be moved to x_1 causing x_1 to have four pebbles. This gives us four distinct vertices that fail to have pebbles, they are v, y_1, y_2, and z_{12}. This means we have at least five "extra" pebbles; i.e., at least one of x_1, x_2 must receive four pebbles and we are done. ∎

Our next theorem tells us that if $|E(G)|$ is large enough then $f(G) = n$.

Theorem 5. *Let G be a connected graph with $n \geq 4$ vertices and q edges. If $q \geq \binom{n-1}{2} + 2$, then $f(G) = n$.*

Proof. One can easily check that G is 2–connected. Our proof will be by induction on n. By inspection we see that our claim is true for $n = 4$. Now suppose our theorem is true for every graph with n' vertices where $4 \leq n' < n$. Let G be a graph on n vertices and let v be our target vertex.

Case (i). Suppose v has degree $n-1$. If v has a pebble on it we are done. Therefore $G - v$ has n pebbles but only $n - 1$ vertices. Hence some vertex, say w, contains two pebbles and we are done.

Case (ii). Suppose v has degree less than $n - 1$. If some vertex adjacent to v has two pebbles on it we are done. Therefore assume that a neighbor of v, say u, has pebble on it. Note that $|E(G - v)| \geq \binom{n-2}{2} + 2$ and $G - v$ is 2–connected. Now the number of pebbles on the vertices of $G - v$ (ignoring the pebble at u) is $n - 1$, so by induction another pebble can be moved to u and we are done.

The last case to consider is that all of the vertices adjacent to v have no pebbles on them. Let w be a neighbor of v. If the degree of w is less than $n - 1$ then by induction we can move a pebble to v in $G - w$. Hence assume $\deg(w) = n - 1$ and note that there are n pebbles among the vertices of $G - w - v$. By the 2–connectivity of G (and the fact that we are assuming no neighbor of v has a pebble) we see that $G - w - v$ contains a vertex without pebbles. Therefore $G - w - v$ contains at least two vertices with 2 or more pebbles each or else some vertex with 4 pebbles. In either case 2 pebbles can be moved to w, hence one pebble can be moved to v and we are done. ∎

To see that Theorem 5 is best possible just take K_{n-1} and adjoin to it some vertex v by a single edge. Call this new graph G. Since G has a cut vertex we must have $f(G) \geq n + 1$.

It would be interesting to know the fewest possible edges a graph G could have with $f(G) = n$. To this effect we mention one interesting result.

Fact 3. Let G be a connected graph on $n \geq 6$ vertices. If $u, v \in V(G)$, $\deg(u) = \deg(v) = 2$, and $\text{dist}(u, v) \geq 3$ then $f(G) \geq n + 1$.

Proof. Let u and v be as mentioned above. Let x_1, x_2 be the neighbors of u and let y_1, y_2 be the neighbors of v. Place one pebble at every vertex of $G - \{u, v, x_1, x_2, y_1, y_2\}$, and six pebbles at u. It is easy to see that no pebbles can be moved to u. □

4. Efficient graphs.

Instead of pebbles, suppose we had a substance that was divisible into infinitesimal amounts. Call such a substance sand. Let G be any connected graph and let $D = diam(G)$. Suppose we place our sand (in piles) at the vertices of G and that as we move sand across an edge (i.e., move one unit distance) we lose half the sand we started with. Then no matter how we distribute 2^D units of sand among the vertices of G it is easy to see that we can move "one unit" of sand to any vertex of G.

Now let's modify our problem. Let ε be of the form $\frac{1}{t}$ (i.e., $\varepsilon = \frac{1}{t}$, $t \in \mathbb{Z}^+$) and place 2^D units of sand randomly on the vertices of G. Here we require the piles of sand to be in multiples of ε. A move now consists of moving $2 \cdot \varepsilon$ units of sand at a time with half of the sand lost as an edge is transversed. It is not at all clear for which graphs G it is possible to move "one unit" of sand to any vertex in G. We can rephrase this problem in terms of pebbling. As before, let G be a connected graph with diameter D; let t be the smallest positive integer such that no matter how $t2^D$ pebbles are distributed among the vertices of G, at least t pebbles can be moved to any vertex in G, then we define $\varepsilon_G = \frac{1}{t}$. If no such t exists we define $\varepsilon_G = 0$. Note $\varepsilon_{Q^n} = 1$.

From physics, we know that energy comes in quanta (discrete "packets"). This motivates our next definition.

Definition. Let G be a connected graph. If $\varepsilon_G > 0$, then we call G an *efficient graph*. That is, if 2^D units of energy are distributed among the vertices of G and if it comes in small enough quanta (ε_G), then one unit of energy can be moved to any desired vertex. A graph with $\varepsilon_G = 1$ is called a *very efficient graph*.

Two questions naturally arise, they are:

Question 1. Can one characterize those graphs G with $\varepsilon_G > 0$?

and

Question 2. For every ε of the form $\varepsilon = \frac{1}{t}$ does there exist a graph G with $\varepsilon_G = \varepsilon$?

We answer both of those questions below.

Definition. Let G be a connected graph and let v, $v' \in V(G)$. If $\text{dist}(v, v') = \text{diam}(G)$ then we will call the vertices v and v' *antipodal vertices*.

Definition. Let G be a connected graph and let $v \in V(G)$. Let $m = \max\{\text{dist}(v, y), y \in V(G) - \{v\}\}$. Clearly, every vertex $y \in G$ lies on a $v - y$ path P_{vy} with $|V(P_{vy})| \leq m + 1$. A *path covering of G rooted at v*, denoted by $\mathcal{P}_v$, is a smallest collection of paths $\{P_1, P_2, \ldots, P_t\}$ such that $P_i = vy_{i_1}y_{i_2}\ldots y_{i_k}$ $(k \leq m)$ and given any $y \in V(G)$ there exists some $P_j \in \mathcal{P}_v$ with $y \in V(P_j)$. (Note that if $P_i = vy_{i_1}y_{i_2}\ldots y_{i_k}$ then $\text{dist}(v, y_{i_j}) = j$.)

Definition (The Antipodal Property). We say that a connected graph G has the *antipodal property* if given any $v, v' \in V(G)$ such that $\text{dist}(v, v') = \text{diam}(G)$ then for any $y \in V(G) - \{v, v'\}$, y lies on a $v - v'$ path P with $|V(P)| = \text{diam}(G) + 1$.

We can now characterize those graphs G with $\varepsilon_G > 0$.

Theorem 6. *Let G be a connected graph. Then G is an efficient graph (i.e. $\varepsilon_G > 0$) if and only if G has the antipodal property.*

Proof. Assume that $\text{diam}(G) = D$. First, we prove necessity. Suppose G fails to have the antipodal property. Then there exist antipodal vertices v and v' and a vertex u such that u fails to lie on a $v - v'$ path P with $|V(P)| = \text{diam}\,(G) + 1$. Let t be any positive integer. Distribute $t2^D$ pebbles on the vertices of G by placing one pebble at u and the remaining $t2^D - 1$ pebbles at v'. Then it is easy to see that at most $t - 1$ pebbles can be moved to v and we are done. Now assume that G has the antipodal property, and let $\hat{f}(u)$ denote the number of pebbles at the vertex $u (u \in V(G))$. Let $\mathcal{A}$ be the set of all antipodal vertices in G. For $v \in \mathcal{A}$ let $\mathcal{P}_v$ be a path covering of $G - v'$ rooted at v. (Here v' is the unique vertex of $V(G)$ such that $\text{dist}(v, v') = D$.) For all other vertices $v \in V(G) - \mathcal{A}$, let $\mathcal{P}_v$ be a path covering of G rooted at v. Let $t = \max\{|\mathcal{P}_v| : v \in V(G)\}$. We will show that $\varepsilon_G \geq \frac{1}{2t}$.

Case (i) $v \in V(G) - \mathcal{A}$. Suppose our target vertex v is in $V(G) - \mathcal{A}$. Then $|\mathcal{P}_v| \leq t$ and each $P \in \mathcal{P}_v$ has length $\leq D - 1$. Let $\mathcal{P}_v = \{P_1, P_2, \ldots, P_j\}$ $(j \leq t)$ and assume that P_i has $m_i 2^{D-1} + r_i$ $(0 \leq r_i < 2^{D-1})$ pebbles on it. (Note that if some vertex u is in several of the paths of $\mathcal{P}_v$, say $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$, we arbitrarily partition the pebbles at u so that $\hat{f}(u) = p^u_{i_1} + p^u_{i_2} + \ldots + p^u_{i_k}$ and think of $p^u_{i_j}$ of the pebbles at u as belonging to the path P_{i_j} but still positioned at the vertex u.) Clearly $\sum_{i=1}^{j} r_i < t2^{D-1}$, hence $\sum_{i=1}^{j} m_i \geq 3t$ (we have $4t \cdot 2^{D-1} = 2t \cdot 2^D$ pebbles to work with). Hence we are done since the pebbling number of each path $P_j \in \mathcal{P}_v$ is at most 2^{D-1}, thus we can move at least $3t$ pebbles to v.

Case (ii) $v \in \mathcal{A}$. Suppose our target vertex $v \in \mathcal{A}$. Let $\mathcal{P}_v = \{P_1, P_2, \ldots P_j\}$ $(j \leq t)$ and let v' be v's antipodal vertex. Suppose $\hat{f}(v') \leq t2^D$ (i.e. half or less of the pebbles are placed at v'), and assume that P_i $(1 \leq i \leq j)$ has $m_i 2^{D-1} + r_i$ $(0 \leq r_i < 2^{D-1})$ pebbles on it. Then by moving pebbles from v' to its neighbors we can be assured that the vertices in $G - v'$ will have at least $3t2^{D-1}$ pebbles and we can argue as before. Therefore, we may assume that $\hat{f}(v') > t2^D$.

Since G has the antipodal property we may assume that $\text{diam}(P_i) = D - 1$ for $1 \leq i \leq j$. For each i $(1 \leq i \leq j)$ move $(2^{D-1} - r_i)$ pebbles from v' to P_i. Now we can move j pebbles to v leaving $m_i 2^{D-1}$ pebbles on P_i $(1 \leq i \leq j)$ and at least $\hat{f}(v') - j2^D$ pebbles at v'. Note that $\left(\hat{f}(v') - j2^D + \left(\sum_{i=1}^{j} m_i\right)(2^{D-1}) \geq t2^D + (t - j)2^D\right)$.

Let $\sum_{i=1}^{j} m_i = k$. Then we can move k more pebbles to v leaving us with at least

$\hat{f}(v') - j2^D \geq t2^D + (t-j)2^D - k2^{D-1}$ pebbles at v' and $j+k$ pebbles at v. If $j+k \leq t$ then v' has at least $t2^D + (t-(j+k))2^D$ pebbles that haven't been moved and we are done.

If $t < j+k < 2t$ then v' has at least $(t-i)2^D$ pebbles (where $i+t = j+k$) that haven't been moved and we are done. ∎

Next, we answer question 2 above.

Theorem 7. *Let $G = K_n - e$. Then $\varepsilon_G = \frac{1}{t}$ where t is the smallest integer such that $2t \geq (n-2)$.*

Proof. We do not explicitly state the pebbling moves - these should be obvious to the reader since $\text{diam}(G) = 2$. Let $V(G) = \{x_1, x_2, \ldots, x_n\}$ and let $e = (x_{n-1}, x_n)$. First we prove that $2t$ cannot be less than $n-2$. Assume that $2t < n-2$. Place one pebble at x_i for $1 \leq i \leq n-2$ and then place the remaining $4t-(n-2) < 2t$ pebbles at x_{n-1}. It is easy to see that it is impossible to move t pebbles to x_n. Therefore we must have $2t \geq n-2$.

Case (i): Assume $v = x_i$ $(i \leq n-2)$ is our target vertex. By symmetry, we may assume that $v = x_1$. Let $q_i = 2m_i + r_i$ $(0 \leq r_i \leq 1)$ denote the number of pebbles at x_i for $1 \leq i \leq n$. First, assume that $\sum_{i=2}^{n} r_i = n-1 > 2t$. Then we must have $q_1 \geq 1$ since $n-1$ is odd $(2t \geq n-2)$. We know that at least $q_1 + \sum_{i=2}^{n} m_i$ pebbles can be moved to x_1 and that $q_1 + 2\sum_{i=2}^{n} m_i + 2t + 1 = 4t$. Hence $q_1 + 2\sum_{i=2}^{n} m_i = 2t-1 = 2(t-1)+1 \Rightarrow$ at least t pebbles can be moved to x_1 and we are done. Thus we may assume that $\sum_{i=2}^{n} r_i \leq n-2 \leq 2t$. But then $q_1 + 2\sum_{i=2}^{n} m_i = 2t$ which implies that at least t pebbles can be moved to x_1.

Case (ii): $v = x_{n-1}$ or x_n. By symmetry we may assume that $v = x_n$. Let q_i be defined as before. If $\sum_{i=1}^{n-2} m_i = m \geq t - q_n$ we are done, thus assume that $\sum_{i=1}^{n-2} m_i = m < t - q_n$ and let $j = t - (q_n + m)$. Let $\sum_{i=1}^{n-2} r_i = r \leq 2t$. Now $q_{n-1} \geq 2(t-m) - q_n$ and $2t = 2(m + q_n + j)$, so $q_{n-1} \geq 2j$. Thus if $r \geq j$ then we are done. Therefore $r < j$.

Let $k = t - (m + q_n + r) \geq 1$ and note that we can move $m + r + q_n$ pebbles to x_n (at "cost" of at most $2m + 3r + q_n$ pebbles) leaving at least $4k$ pebbles at x_{n-1} that haven't been moved. After moving k of these to x_n we are done. ∎

5. Known values of f(G)

Very little is known about the exact value of $f(G)$ except for some trivial cases such as paths ($f(P_n) = 2^n$), complete graphs ($f(K_n) = n$), and the results of Chung [ ] (Chung shows that $f(P_{n_1} \times P_{n_2} \times \ldots \times P_{n_s}) = 2^{n_1+n_2+\cdots+n_s}$ and also gives a method for finding $f(T)$ when T is a tree.) In this section we find $f(G)$ explicitly for several types of graphs. First, we find the pebbling number of odd cycles.

Theorem 8. $f(C_{2k+1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1.$

Proof. Let $C_{2k+1} = xa_{k-1}a_{k-2}\ldots a_2a_1vb_1b_2\ldots b_{k-1}yx$ and let v be our target vertex. Let P_A be the path $va_1a_2\ldots a_{k-2}a_{k-1}$ and let P_B be the path $vb_1b_2\ldots b_{k-2}b_{k-1}$. Let $\hat{f}(u)$ denote the number of pebbles at vertex $u \in V(C_{2k+1})$. Note that $f(P_A) = f(P_B) = 2^{k-1}$. First, we will show necessity. Suppose we are given only $2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor$ pebbles. Then place $\left\lfloor \frac{2^{k+1}}{3} \right\rfloor$ pebbles at x and $\left\lfloor \frac{2^{k+1}}{3} \right\rfloor$ pebbles at y. It is easy to see that at most $2^{k-1} - 1$ pebbles can be moved to a_{k-1} (or b_{k-1}) and we are done.

Now suppose that we have placed $2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$ pebbles at the vertices of C_{2k+1} but that we cannot move a pebble to v. Let $j_A = \sum_{i=1}^{k-1} \hat{f}(a_i)$ and let $j_B = \sum_{i=1}^{k-1} \hat{f}(b_i)$. Then we must have

$$j_A + \left\lfloor \frac{\hat{f}(x) + \left\lfloor \frac{\hat{f}(y)}{2} \right\rfloor}{2} \right\rfloor \leq 2^{k-1} - 1 \qquad (1)$$

and also

$$j_B + \left\lfloor \frac{\hat{f}(y) + \left\lfloor \frac{\hat{f}(x)}{2} \right\rfloor}{2} \right\rfloor \leq 2^{k-1} - 1 \qquad (2)$$

Hence,

$$j_A + j_B + \left\lfloor \frac{\hat{f}(x) + \left\lfloor \frac{\hat{f}(y)}{2} \right\rfloor}{2} \right\rfloor + \left\lfloor \frac{\hat{f}(y) + \left\lfloor \frac{\hat{f}(x)}{2} \right\rfloor}{2} \right\rfloor \leq 2^k - 2 \qquad (3)$$

Equation (1) above results from moving as many pebbles as possible from x and y to a_{k-1} and equation (2) results from moving as many pebbles as possible from x and y to b_{k-1}.

Note that $j_A + j_B + \hat{f}(x) + \hat{f}(y) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$. Thus to minimize the LHS of (3) it is sufficient to assume that $j_A = j_B = 0$ (i.e. we may assume that all the pebbles are at x and y).

Now $2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$ is odd so exactly one of $\hat{f}(x), \hat{f}(y)$ is even. Without loss of generality assume $\hat{f}(x)$ is even.

Now suppose that as many pebbles as possible are moved from x and y to a_{k-1}. When we are done we could still have a pebble left at x and a pebble left at y (i.e.

$\hat{f}(y) \equiv 3 \bmod 4)$ but since $\hat{f}(x)$ is even we may think of both of these pebbles as coming from y. Similarly, if we move as many pebbles as possible from x and y to b_{k-1}, we see that x will have no pebbles ($\hat{f}(x)$ is even) and y will have at most one pebble (i.e. $\hat{f}(x) \equiv 0 \bmod 4$) and we can think of this pebble as originating from y.

Thus from (3) we have $\frac{3}{4}\hat{f}(x) + \frac{3}{4}\hat{f}(y) - \frac{5}{4} \leq 2^k - 2$. (The $-\frac{5}{4}$ comes from the possible pebbles left behind at x and y.) But $\hat{f}(x) + \hat{f}(y) \geq 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1 \geq 2 \left(\frac{2^{k+1}-2}{3}\right) + 1 = \frac{4}{3}(2^k - 1) + 1$. So $\frac{3}{4}\left(\hat{f}(x) + \hat{f}(y)\right) - \frac{5}{4} \geq (2^k - 1) + \frac{3}{4} - \frac{5}{4} = 2^k - \frac{3}{2}$ and this is a contradiction. ∎

Using a similar method of proof as in Theorem 8 it is easy to see that $f(C_{2k}) = 2^k$.

Let $G = (V(G), E(G))$ be a connected graph. Then G^p $(p > 1)$ (the pth power of G) is the graph obtained from G be adding the edge (u, v) to G whenever $2 \leq \operatorname{dist}(u, v) \leq p$ in G. Hence $G^p = (V(G), E(G) \cup \{(u, v);\ 2 \leq \operatorname{dist}(u, v) \leq p \text{ in } G\})$. If $p = 1$, we define $G^1 = G$.

Our next result involves squares of paths. In what follows, we assume $0 \leq r \leq 1$.

Theorem 9. $f(P^2_{2k+r}) = 2^k + r$

Proof. First, we will show that $f(P^2_{2k+1}) \geq 2^k + 1$. Let $P^2_{2k+1} = x_1x_2\ldots x_{2k}x_{2k+1}$ (the edges between x_i and x_{i+2} are implied for $1 \leq i \leq 2k-1$) and place $2^k - 1$ pebbles at x_{2k+1} and one pebble at x_{2k}. It is easy to see that a pebble cannot be moved to x_1, therefore $f(P^2_{2k+1}) \geq 2^k + 1$. Since the diameter of P^2_{2k} is k, we have $f(P^2_{2k}) \geq 2^k$. . As before $\hat{f}(u)$ will denote the number of pebbles at vertex u.

We proceed by induction on $2k + r$. Clearly, our theorem is correct if $2k + r = 1, 2, 3, 4$, or 5.

Case (i) $r = 0$. Suppose that for all $2^{k'} + r$ with $5 \leq 2^{k'} + r < 2k$ we have $f(P^2_{2k'+r}) = 2^{k'} + r$. We will show that $f(P^2_{2k}) = 2^k$. Place 2^k pebbles at the vertices of $P^2_{2k} = x_1x_2\ldots x_{2k}$ and assume (first) that $v \neq x_1$ or x_{2k}.

Note that $\operatorname{dist}(u, x_1) \leq k-1$ for all $u \in V(P^2_{2k}) - \{x_{2k}\}$ and that $\operatorname{dist}(u, x_{2k}) \leq k - 1$ for all $u \in V(P^2_{2k}) - \{x_1\}$. Whence, if $\hat{f}(x_{2k}) \geq 2^{k-1}$ or if $\hat{f}(x_1) \geq 2^{k-1}$ we are done. Thus we may assume that $\hat{f}(x_1) + \hat{f}(x_{2k}) \leq 2^k - 2$. By moving as many pebbles as possible from x_1 to x_2 and from x_{2k} to x_{2k-1} we see that the subgraph $x_2x_3\ldots x_{2k-1} \equiv P^2_{2(k-1)}$ contains at least 2^{k-1} pebbles and we are done by induction.

Therefore, we may assume that $v = x_1$ or x_{2k}. By symmetry, we may assume that $v = x_1$. Suppose that $\sum_{i=2}^{2k-2} \hat{f}(x_i) = p \geq 2$. Then $\hat{f}(x_{2k-1}) + \hat{f}(x_{2k}) = 2^k - p \leq 2^k - 2$ and it is easy to see that pebbles can be moved from x_{2k-1} and x_{2k} so that the subgraph $x_1x_2\ldots x_{2k-2}$ contains at least 2^{k-1} pebbles. Therefore, assume that $\sum_{i=2}^{2k-2} \hat{f}(x_i) \leq 1$. Recall that $\text{dist}(x_{2k-1}, x_1) = \text{dist}(x_{2k-2}, x_1) = k-1$. Thus if $\hat{f}(x_{2k-1}) \geq 2$ then 2^{k-1} pebbles can be moved to x_{2k-1} $(\hat{f}(x_{2k-1}) + \hat{f}(x_{2k}) \geq 2^k - 1)$ and we are done. Therefore, assume that $\hat{f}(x_{2k-1}) \leq 1$ and that $\hat{f}(x_{2k}) \geq 2^k - 2$. If $\sum_{i=2}^{2k-2} \hat{f}(x_i) = 1$, then using the pebbles at x_{2k} we see that at least $2^{k-1} - 1$ pebbles can be moved to x_{2k-2} and we are done $(x_1 x_2 \ldots x_{2k-2} \equiv P^2_{2k-2})$. Whence, we may assume that $\sum_{i=2}^{2k-2} \hat{f}(x_i) = 0 \Rightarrow \hat{f}(x_{2k}) \geq 2^k - 1$ and that $\hat{f}(x_{2k-1}) \leq 1$. This case is also very easy to handle.

Case (ii) $r = 1$. Suppose that for all $2k' + r$ with $5 \leq 2k' + r < 2k + 1$ we have $f(P^2_{2k'+r}) = 2^{k'} + r$. We will show that $f(P^2_{2k+1}) = 2^k + 1$. Place $2^k + 1$ pebbles at the vertices of $P^2_{2k+1} \equiv x_1 x_2 \ldots x_{2k} x_{2k+1}$ $(k \geq 3)$. First, suppose our target vertex $v \neq x_1$ or x_{2k+1}. If $\sum_{i=2}^{2k} \hat{f}(x_i) \geq 3$ then by moving pebbles from x_1 to x_2 and from x_{2k+1} to x_{2k} we see that the subgraph $x_2 x_3 \ldots x_{2k}$ contains at least $2^{k-1} + 1$ pebbles and we are done. Therefore, we may assume that $\sum_{i=2}^{2k} \hat{f}(x_i) \leq 2$. If $\sum_{i=2}^{2k} \hat{f}(x_i) = 2$, then only one of $\hat{f}(x_1)$ and $\hat{f}(x_{2k+1})$ can be odd - this case is also easy to handle.

Therefore, assume that $\sum_{i=2}^{2k} \hat{f}(x_i) \leq 1$. If $v \neq x_2, x_{2k}$ then we can move at least $2^{k-1} - 1$ pebbles to $x_3 x_4 \ldots x_{2k-1} \equiv P^2_{2(k-2)+1}$ and we are done. Thus assume (by symmetry) that $v = x_2$ and that $\sum_{i=2}^{2k} \hat{f}(x_i) \leq 1$. If $\hat{f}(x_1) \geq 2$ we are done. Hence assume that $\hat{f}(x_1) \leq 1$. If $\sum_{i=2}^{2k} \hat{f}(x_i) = 0$ then $\hat{f}(x_{2k}) \geq 2^k$ and we are done. Thus, assume that $\sum_{i=2}^{2k} \hat{f}(x_i) = 1$. Now $\hat{f}(x_{2k+1}) \geq 2^k - 1$ which implies that $2^{k-1} - 1$ pebbles can be moved to x_{2k-1} (from x_{2k+1}) and we are done $(x_2 x_3 \ldots x_{2k-1} \equiv P^2_{2(k-1)})$.

Therefore, we may assume (by symmetry) that $v = x_1$. Arguing as before, we see that if $\sum_{i=2}^{2k-1} \hat{f}(x_i) \geq 2$ then we are done. (If $\sum_{i=2}^{2k-1} f(x_i) = 2$ use the fact that one of $\hat{f}(x_{2k})$ or $\hat{f}(x_{2k+1})$ is even.) Thus, we have $\sum_{i=2}^{2k-1} \hat{f}(x_i) = 1$ (If $\sum_{i=2}^{2k-1} \hat{f}(x_i) = 0$ then move 2^{k-1} pebbles to x_{2k-1} and use the fact that $\text{dist}(x_{2k-1}, x_1) = k-1$.) Let x_j be the unique vertex in $\{x_2, \ldots, x_{2k-1}\}$ that contains a pebble and note that

$2^{k-1}-1$ pebbles can be moved to x_{2k-1} (from x_{2k} and x_{2k+1}). If j is odd we are done since $x_1x_3x_5\ldots x_j\ldots x_{2k-1}\equiv P_k$ and $f(P_k)=2^k$.

So we may assume that j is even and that $\text{dist}(x_j,x_{2k})=i$. Let $\hat{f}(x_{2k})=2^im+q$ where $0\le q<2^i$. First, assume m is odd and note that $\hat{f}(x_{2k+1})\ge 2^k-2^i(m+1)$ $=2^{i+1}(2^l-\frac{(m+1)}{2})$ (here $i+1+l=k$). This tells us that we can move $2^l-\frac{(m+1)}{2}$ pebbles from x_{2k+1} to x_{j-1} ($\text{dist}(x_{2k+1},x_{j-1})=i+1$). We can then move m pebbles from x_{2k} to x_j giving us $m+1$ pebbles at x_j ($\hat{f}(x_j)=1$). Hence, we can move an additional $\left(\frac{m+1}{2}\right)$ pebbles to x_{j-1}. This gives us a total of 2^l pebbles at x_{j-1} and we are done ($\text{dist}(x_1,x_{j-1})=l$).

Now assume that m is even. First move 2^i-q pebbles to x_{2k} from x_{2k+1}. This leaves at least $2^k-2^i(m+2)=2^{i+1}(2^l-\frac{(m+2)}{2})$ pebbles at x_{2k+1}. Move $2^l-\frac{(m+2)}{2}$ of these pebbles to x_{j-1} (from x_{2k+1}). Now move $m+1$ pebbles to x_j (from x_{2k}) and note that x_j now contains $m+2$ pebbles ($\hat{f}(x_j)=1$). Arguing as before, we see that 2^l pebbles can be moved to x_{j-1} and we are done. ∎

We know that if p is large enough (i.e. $p\ge n-1$) then $G^p\equiv K_n$. Define the *pebbling exponent* of a graph G (denoted by p_G) to be the least power p such that $f(G^p)=n$.

Question 3. What is p_G when G is a cycle?

6. Optimal Pebbling

Consider the problem of placing pebbles at the vertices of a graph G so that a pebble can be moved to any desired vertex, and as few pebbles are used as possible. Such a pebbling of G is called an *optimal pebbling* and the number of pebbles used is called the *optimal pebbling number* of G (denoted by $of(G)$). Finding the optimal pebbling number of an abritrary graph G appears to be more difficult than finding its pebbling number; $of(Q^n)$ is unknown and to find $of(P_n)$ requires a little work as demonstrated below.

In what follows, we assume that $0\le r\le 2$.

Theorem 10. *The optimal pebbling number of P_{3t+r} is $2t+r$.*

First some remarks on notation. Let $P_{3t+r}\equiv x_1x_2\ldots x_{3t+r}$. Let $\mathbf{P}$ be an optimal pebbling of P_{3t+r} and let $f_{\mathbf{P}}(x_i)$ denote the number of pebbles at x_i. Suppose that $f_{\mathbf{P}}(x_j)=0$ and that $i<j$, then we say that *vertex x_j can be reached from x_i* if the subpath $x_ix_{i+1}\ldots x_j$ has enough pebbles so that a pebble can be moved to x_j. If $f_{\mathbf{P}}(x_j)>0$ and $i<j$, then we say that vertex x_j can be reached from x_i if two pebbles can be moved to x_{j-1} in the subpath $x_ix_{i+1}\ldots x_{j-1}$. A similar definition applies if $i>j$.

We will use the notation x_i' to designate the fact that $f_{\mathbf{P}}(x_i) \geq 1$ (i.e., that x_i has a pebble). The notation $x_{i_0}^*$ will be used to denote the vertex closest to x_{3t+r} such that x_{3t+r} can be reached from x_{i_0}. Thus $x_1 x_2 \ldots x_{j_1-1} x_{j_1}' x_{j_1+1}' \ldots x_{k_1}' x_{k_1+1} \ldots x_{j_2-1} x_{j_2}' \ldots x_{k_2}' x_{k_2+1} \ldots x_{i_0}^* \ldots x_{j_s-1} x_{j_s}' \ldots x_{k_s}' x_{k_s+1} \ldots x_{3t+r}$ (1) denotes the fact that $f_{\mathbf{P}}(x_j) \geq 1$ if and only if j is in one of the following intervals $[j_1, k_1]$, $[j_2, k_2]$, $\ldots$, $[j_s, k_s]$, and the fact that x_{3t+r} can be reached from x_{i_0} but not by any x_j with $j > i_0$. We call (1) the pebbling scheme of $\mathbf{P}$, the subpaths $[j_i, k_i] = x_{j_i}' x_{j_i+1}' \ldots x_{k_i}'$ ($1 \leq i \leq s$) *closed intervals* and the subpaths $(1, j_1-1) = x_1 x_2 \ldots x_{j_1-1}$, $(k_i+1, j_{i+1}-1) = x_{k_i+1} \ldots x_{j_{i+1}-1}$ ($1 \leq i \leq s-1$), $(k_s+1, 3t+r) = x_{k_s+1} \ldots x_{3t+r}$ *open intervals*. Hence, the symbol $[j_i, k_i]$ will refer to either a collection of integers or else a subpath of P_{3t+r}, context should make it clear. Thus (1) can be rewritten as $(1, j_1-1)[j_1, k_1](k_1+1, j_2-1) \ldots x_{i_0}^* \ldots [j_s, k_s](k_s+1, 3t+r)$.

Now for a proof of our theorem.

Proof of Theorem 10. First, we will show that $of(P_{3t+r}) \leq 2t + r$. It is easy to check that our theorem is true when $3t+r = 1, 2, 3, 4$, or 5. Therefore, assume that $3t + r \geq 6$, let $P_{3t+r} = x_1 x_2 \ldots x_{3t+r}$ and place 2 pebbles at $x_2, x_{3+2}, \ldots, x_{3(t-1)+2}$. We are done if $r = 0$. If $r = 1$ place an additional pebble at x_{3t+1} and if $r = 2$ place another additional pebble at x_{3t+2} and we are done.

Our proof will be by induction on $3t + r$. Suppose at some point our theorem is false and let $3t + r$ be the least number where this occurs. Clearly, we must have $of(P_{3t+r}) \geq of(P_{3t+r-1})$. This tells us that $r \neq 0$.

Case (i) $r = 1$. Suppose $r = 1$. Then we must have $of(P_{3t+1}) = 2t = of(P_{3t})$. Let $\mathbf{P}$ be an optimal pebbling of P_{3t+1}. Clearly, $f_{\mathbf{P}}(x_{3t+1}) = 0$, otherwise we would have $of(P_{3t}) < 2t$ - a contradiction. Furthermore, we claim that $f_{\mathbf{P}}(x_{3t}) \leq 1$. Suppose $f_{\mathbf{P}}(x_{3t}) \geq 2$, then by moving as many pebbles as possible from x_{3t} to x_{3t-1} we get an optimal pebbling of P_{3t-1} that uses at most $2t - 1$ pebbles - a contradiction. Therefore $f_{\mathbf{P}}(x_{3t}) \leq 1$. Let $\mathcal{P} = \{\mathbf{P}_1, \mathbf{P}_2, \ldots\}$ be the collection of all optimal pebblings of P_{3t+1}. Let $\mathbf{P}_i \in \mathcal{P}$ and let $a_j^i = \sum_{k=3t+2-j}^{3t+1} f_{\mathbf{P}_i}(x_k)$ for $1 \leq j \leq 3t+1$ and let $A(\mathbf{P}_i) = (a_1^i, a_2^i, \ldots, a_{3t+1}^i)$. We will use the $A(\mathbf{P}_i)$'s to order the elements of $\mathcal{P}$. Consider $\mathbf{P}_i, \mathbf{P}_j \in \mathcal{P}$ and let k be the least integer such that $a_k^i \neq a_k^j$. If $a_k^i > a_k^j$ then we say that $\mathbf{P}_i > \mathbf{P}_j$ (otherwise $\mathbf{P}_j > \mathbf{P}_i$). Now let $\mathbf{P} \in \mathcal{P}$ be the greatest such optimal pebbling of P_{3t+1} under this ordering and let $(1, j_1-1)[j_1, k_1] \ldots x_{i_0}^* \ldots [j_s, k_s](k_s+1, 3t+1)$ be its pebbling scheme.

Suppose $x_{i_0}^* \in [j_s, k_s]$, then by our choice of $\mathbf{P}$ we see that $f_{\mathbf{P}}(x_{k_s}) \geq 2$, hence $k_s \leq 3t - 1$. But then some vertex in $[j_s, k_s]$ must contain 3 or more pebbles since $f_{\mathbf{P}}(x_{3t}) = f_{\mathbf{P}}(x_{3t+1}) = 0$. Let $j \in [j_s, k_s]$ be the largest integer such that $f_{\mathbf{P}}(x_j) \geq 3$. We can create a new optimal pebbling $\mathbf{P}'$ of P_{3t+1} by letting $f_{\mathbf{P}'}(x_i) = f_{\mathbf{P}}(x_i)$ for $1 \leq i \leq 3t+1$ ($i \neq j-1, j, j+1$) and by letting $f_{\mathbf{P}'}(x_{j-1}) = f_{\mathbf{P}}(x_{j-1}) + 1$, $f_{\mathbf{P}'}(x_j) = f_{\mathbf{P}}(x_j) - 2$, and $f_{\mathbf{P}'}(x_{j+1}) = f_{\mathbf{P}}(x_{j+1}) + 1$. A little thought should convince the reader that "all" vertices that could be reached using pebbles from $[j_s, k_s]$ in $\mathbf{P}$ can still be reached in $\mathbf{P}'$.

But now we have $\mathbf{P}' > \mathbf{P}$ - a contradiction. Therefore $x_{i_0}^* \notin [j_s, k_s]$. Let

$x^*_{i_0} \in [j_i, k_i]$ $(i < s)$. By our definition of $x^*_{i_0}$ we see that some vertex in $[j_i, k_i]$ must contain at least three pebbles. Let j be the largest integer in $[j_i, k_i]$ such that $f_{\mathbf{P}}(x_j) \geq 3$. Now arguing as before we see that there exists another optimal pebbling $\mathbf{P}'$ of P_{3t+1} such that $\mathbf{P}' > \mathbf{P}$ and we are done.

Case (ii) $r = 2$. This case is similar to the previous case and will be omitted. ∎

After examing the optimal pebbling number for all trees with 7 vertices or less, the following conjecture seems reasonable.

Conjecture. Let $\mathcal{T}_{(n,i)}$ be the collection of all trees on n vertices with i vertices of degree one. Let $t_{(n,i)} = |\mathcal{T}_{(n,i)}|$ and let $a_{(n,i)} = \dfrac{\sum_{T \in \mathcal{T}_{(n,i)}} of(T)}{t_{(n,i)}}$. Then $a_{(n,2)} \geq a_{(n,3)} \geq \cdots \geq a_{(n,n-1)}$.

What about $of(Q^n)$? We will show that if $n = 2k+1$ then $of(Q^n) \leq 2^{k+1}$ and if $n = 2k$ then $of(Q^n) \leq 2^k + 2^{k-1}$. It is best to think of the vertices of Q^n as subsets of an n-element set. If $n = 2k+1$, then place 2^k pebbles at $[n] = \{1, 2, \ldots, n\}$ and 2^k pebbles at . If $n = 2k$, then place 2^k pebbles at $[n]$ and 2^{k-1} pebbles at . In either case it is easy to see that a pebble can be moved to any vertex v.

Problem. Find $of(Q^n)$.

7. Conclusion

Probably, the most interesting conjecure about pebbling graphs is the following.

Conjecture. (Graham) $f(G \times H) \leq f(G) \cdot f(H)$.

See [1], [4] and [5] for partial results pertaining to this conjecture.

References

[1] F.R. K. Chung, *Pebbling in hypercubes*, SIAM J. Disc. Math Vol. 2, No. 4 (1989), pp 467-472.

[2] D. Duffus and I. Rival, *Graphs orientable as distributive lattices*, Proc. Amer. Math. Soc. 88 (1983), pp 197-200.

[3] P. Lemke and D. Kleitman, *An additional theorem on the integers modulo n*, J. Number Theory 31 (1989), pp 335-345.

[4] D. Moews, *Pebbling graphs*, J. of Combinatorial Theory (Series B) 55 (1992), pp 244-252.

[5] H. S. Snevily and J. Foster, *The 2-Pebbling Property and a Conjecture of Graham's*, submitted.