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OPTIMIZING BULL-FREE PERFECT GRAPHS*

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Abstract. A bull is a graph with five vertices a, b, c, d, e and five edges ab, ac, bc, da, eb. Here we present polynomial-time combinatorial algorithms for the optimal weighted coloring and weighted clique problems in bull-free perfect graphs. The algorithms are based on a structural analysis and decomposition of bull-free perfect graphs.

Key words. graph algorithms, perfect graphs, analysis of algorithms and problem complexity, combinatorial optimization

AMS subject classifications. 05C85, 05C60, 68Q25, 90C27

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1. Introduction.

A graph G is called *perfect* if the vertices of every induced subgraph G' of G can be colored with $\omega(G')$ colors, where $\omega(G')$ is the maximum clique size in H. Berge [1] introduced perfect graphs and conjectured the following characterization: *A graph is perfect if and only if it contains no odd hole and no odd antihole as an induced subgraph,* where a *hole* is a chordless cycle with at least five vertices, and an *antihole* is the complement of a hole. Graphs with no odd hole and no odd antihole have become known as *Berge* graphs. This conjecture, known as the strong perfect graph conjecture, was proved recently by Chudnovsky et al. [5]; thus *every Berge graph is perfect.* One problem that is not yet solved in this context is the existence of a combinatorial algorithm to compute the chromatic number of a perfect graph. Here we will give such an algorithm for bull-free Berge graphs, i.e., graphs with no induced subgraph isomorphic to a bull, where a *bull* is a graph with five vertices a, b, c, d, e and five edges ab, bc, cd, be, ce (see Figure 1). Our algorithm is based on specific properties of these graphs. Let us recall that Chvátal and Sbihi [3] proved the validity of the strong perfect graph conjecture for bull-free graphs, and subsequently Reed and Sbihi [18] gave a polynomial algorithm for recognizing bull-free Berge graphs.

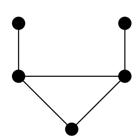

FIG. 1. *The bull.*

In this paper, we present polynomial-time algorithms for solving the following optimization problems for bull-free perfect graphs: find a largest clique, a largest stable

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set, a minimum coloring, and a minimum clique covering. We actually present algorithms which solve the weighted versions of these problems, defined as follows. We are given a graph G with vertices $v_1, \ldots, v_n$ and positive integer weights $w(v_1), \ldots, w(v_n)$.

Maximum weighted clique problem. Find a clique K of G, such that the weight of K, defined as the sum of the weights of the vertices of K, $w(K) = \sum_{x \in K} w(x)$, is maximum over all cliques of G.

Maximum weighted stable set problem. Find a stable set S of G, such that the weight of S, defined as the sum of the weights of the vertices of S, $w(S) = \sum_{x \in S} w(x)$, is maximum over all stable sets of G.

Minimum weighted coloring problem. Find stable sets $S_1, \ldots, S_t$ and integers $W(S_1), \ldots, W(S_t)$, such that

$$\sum_{S_i \ni v_j} W(S_i) \geq w(v_j) \qquad (\forall v_j) \tag{1}$$

and the sum $W(S_1) + \cdots + W(S_t)$ is minimum over all sets of integers that satisfy (1).

Minimum weighted clique covering problem. Find cliques $K_1, \ldots, K_t$ and weights $W(K_1), \ldots, W(K_t)$, such that

$$\sum_{K_i \ni v_j} W(K_i) \geq w(v_j) \qquad (\forall v_j) \tag{2}$$

and the sum $W(K_1) + \cdots + W(K_t)$ is minimum over all sets of integers that satisfy (2).

Recall that if G is a perfect graph, classical polyhedral considerations (see [12]) imply that (a) the optimal value of the maximum weighted clique problem and of the minimum weighted coloring problem are equal; (b) there exists a minimum weighted coloring that satisfies (1) with equality for every vertex. The same facts hold for the maximum weighted stable set problem and the minimum weighted clique covering problem.

It is possible to color every perfect graph optimally and in polynomial time, thanks to the algorithm of Grötschel, Lovász, and Schrijver [12]; but that algorithm is based on the ellipsoid method and may be rather complex and impractical. In contrast, the algorithm we are going to present here exploits the combinatorial structure of bull-free graphs and is fairly transparent. We will find it convenient, however, to use the following argument. Let $\mathcal{C}$ be a self-complementary class of perfect graphs. *If there exists a strongly polynomial-time algorithm A that can compute the weighted clique number of any graph G in $\mathcal{C}$ in time $O(n^k)$ (n being the number of vertices of G), then there exists a strongly polynomial-time algorithm A' that can construct a minimum weighted coloring for any graph G in $\mathcal{C}$ in time $O(n^{k+4})$.* This argument is implicit in [12, section 9.4] and in [19, Proof of Corollary 67.5c and Theorem 67.6], and we do not copy its proof here. It suffices to note that A' consists mainly in at most n^4 calls to A applied to weighted subgraphs of G and $\overline{G}$; this is independent of the method that A is based on. Since the class of bull-free Berge graphs that we consider here is self-complementary, this argument can be applied; this allows us, therefore, to focus on only one of the above four problems, namely, the maximum weighted clique number.

Roughly speaking, our algorithm follows a decomposition procedure for bull-free Berge graphs; with each bull-free Berge graph G a decomposition tree is associated;

our algorithm uses some known polynomial-time algorithms to solve the problem for the leaves of the tree (these indecomposable graphs turn out to belong to well-known classical families); it then recursively combines solutions along the tree, upward from children to parent, up to the root G. A key point in our proofs is the use of decomposition theorems in order to show how to combine the solutions properly from the children to the parent. Another key point is to show that the number of tree nodes is polynomial so that the total running time of our algorithm itself is polynomial.

In order to present this algorithm exactly and to justify it, a number of definitions and results must be recalled; also, some new results will be proved. The algorithm will be described precisely in section 6.

2. Definitions. Apart from standard graph-theoretic terms, we use the verbs "see" and "miss" instead of "be adjacent to" and "not be adjacent to." The neighborhood $N(x)$ of a vertex x in a graph G is the set of all vertices of $G \setminus x$ that see x. A chordless path on k vertices is denoted by P_k. Unless otherwise specified, the phrase "G contains H" means "G contains H as an induced subgraph." Note also that a graph G is bull-free if and only if its complement $\overline{G}$ is bull-free. For any subset X of vertices of a graph G, we let $G[X]$ denote the subgraph of G induced by X.

Weakly triangulated graphs. A graph is called *weakly triangulated* if it does not contain a hole or an antihole. Hayward [13] proved that all weakly triangulated graphs are perfect. Subsequently, Hayward, Hoàng, and Maffray [14] gave polynomial-time algorithms that solve the four optimization problems above for weakly triangulated graphs.

Transitively orientable graphs. A graph is called *transitively orientable* if it admits a *transitive orientation*, i.e., an orientation of its edges with no circuit and with no P_3 abc with the orientation $\vec{ab}$ and $\vec{bc}$. Such graphs are also called *comparability graphs.* A well-known subclass of comparability graphs is the class of P_4-free graphs, also called *cographs* [8]. Indeed, a result Seinsche [20] states is that for every P_4-free graph G on at least two vertices, either G or its complement $\overline{G}$ is disconnected; from this it is easy to derive that every P_4-free graph is transitively orientable.

Partial vertices, homogeneous sets. Given a subset of vertices S in a graph G, a vertex from $G \setminus S$ is *partial on* S or S*-partial* if it has at least one neighbor and at least one nonneighbor in S. A vertex from $G \setminus S$ is *impartial on* S if it either sees all vertices of S or misses all vertices of S.

A *homogeneous set* (or *module*) in a graph $G = (V, E)$ is a subset $S \subseteq V$ such that every vertex from $G \setminus S$ sees either all or none of S. A homogeneous set S is *proper* if $2 \leq |S| \leq |V| - 1$. Note that if S is a homogeneous set of G, then it is also a homogeneous set of the complementary graph $\overline{G}$. A graph is called *prime* if it has no proper homogeneous set.

Homogeneous pairs. A *homogeneous pair* [3] in a graph G is a pair of disjoint subsets of vertices Q_1, Q_2 such that all Q_1-partial vertices are in Q_2; all Q_2-partial vertices are in Q_1; at least one of Q_1, Q_2 includes at least two vertices; and there are at least two vertices in $G \setminus Q_1 \cup Q_2$. Note that if Q_1, Q_2 is a homogeneous pair in G, then it is also a homogeneous pair in $\overline{G}$.

Whenever a graph G admits a homogeneous pair Q_1, Q_2, we will denote by T_i the set of vertices of $G \setminus (Q_1 \cup Q_2)$ that see all of Q_i and miss all of Q_{3-i} $(i = 1, 2)$, by T the set of vertices of $G \setminus (Q_1 \cup Q_2)$ that see all of $Q_1 \cup Q_2$, and by Z the set of vertices of $G \setminus (Q_1 \cup Q_2)$ that miss all of $Q_1 \cup Q_2$. Following [3], we may decompose G along this homogeneous pair into two graphs H and Q defined as follows. The graph H is made from $G \setminus (Q_1 \cup Q_2)$ by adding four vertices u_1, u_2, s_1, s_2 with edges u_1s_1, u_2s_2,

u_1s_2, u_2s_1, s_1s_2 and with edges tu_i, ts_i for every vertex $t \in T_i \cup T$ for each $i = 1, 2$. The graph Q is the subgraph of G induced by $Q_1 \cup Q_2$.

Let us say that a homogeneous pair Q_1, Q_2 is *interesting* if both Q_1, Q_2 induce connected subgraphs of G, $Q_1 \cup Q_2$ contains a square with an edge in Q_1 and an edge in Q_2, $T_1 \neq \emptyset$, $T_2 \neq \emptyset$, and there exists an edge t_1t_2 with $t_1 \in T_1$, $t_2 \in T_2$.

In [7] the following result was proved (although not stated explicitly this way).

THEOREM 2.1. *Let G be a prime bull-free Berge graph G. If G contains an even hole, then G admits a "box partition."*

The *box partition* is a structural concept whose exact definition we defer to section 3. The proof of that theorem in [7] is actually a polynomial-time algorithm which, given a bull-free Berge graph G, produces a proper homogeneous set of G, or asserts that G contains no even hole, or produces a box partition. Our interest in the box partition here is due mainly to the following lemma.

LEMMA 2.2 (the transitive box partition lemma). *Let G be a bull-free Berge graph with no antihole. If G has a box partition, then G admits a transitive orientation.*

This lemma will be proved in section 3.

THEOREM 2.3. *Let G be a prime bull-free Berge graph that contains a hole and an antihole. Then the following hold.*

(I) *The graph G contains an interesting homogeneous pair Q_1, Q_2.*

(II) *If H, Q are the two graphs obtained by decomposing G along an interesting homogeneous pair, then both H and Q are bull-free Berge graphs.*

(III) *It is possible to build a solution of the maximum weighted clique problem on G from a solution of the same problem on H and Q with appropriately defined vertex-weights.*

This theorem will be proved in section 5.

3. Boxes and transitive orientations. For any subset B of vertices in a graph G, we let $M(B)$ denote the set of vertices of $G \setminus B$ that are partial on B.

DEFINITION 3.1 (the box partition). *Let G be a graph with vertex set V. We call box partition any partition of V into disjoint nonempty subsets called the boxes, inducing connected subgraphs which satisfy the following properties:*

(i) *Each box is labeled either "odd" or "even" (each vertex will be labeled odd or even accordingly), and there is no edge between two odd boxes or between two even boxes.*

(ii) *For each box B such that $M(B) \neq \emptyset$, there exist in $V - B$ two auxiliary adjacent vertices a_B and a'_B, such that a_B sees all of B and misses all of $M(B)$, while a'_B sees all of $M(B)$ and misses all of B.*

Remark 1. When G is bull-free, the fact that a'_B sees every vertex of $M(B)$ is a consequence of the other facts given in property (ii).

Indeed, if a'_B missed a vertex x of $M(B)$, then there should exist adjacent vertices u, v in B such that x sees u and misses v, and then a_B, u, v, x, a'_B would be a bull.

Let us note that if a bull-free perfect graph G with no proper homogeneous set and no $\overline{C}_6$ admits a box partition, then two further properties hold. Say that two neighborhoods $N(u), N(v)$ are *comparable* if $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$ holds.

(iii) *Every box is P_4-free.*

(iv) *Any two adjacent vertices in B have comparable neighborhoods in $M(B)$.*

To prove (iii), we recall that a *broom* is the graph made up of a P_4, plus a fifth vertex adjacent to all vertices of the P_4, plus a sixth vertex adjacent to the fifth vertex only. We proved the following result.

LEMMA 3.2 (the broom lemma [7]). *If a bull-free, C_5-free graph contains a broom, then it has a proper homogeneous set which contains the P_4 of the broom.*

Now observe that if a box B contains a P_4, then adding the vertices a_B and a'_B we obtain a broom, and then by the broom lemma G should contain a proper homogeneous set, which is a contradiction.

To prove (iv), suppose on the contrary that some two adjacent vertices u, v in a box B have incomparable neighborhoods in $M(B)$. So there exist a vertex x in $M(B) \cap N(u) - N(v)$ and a vertex y in $M(B) \cap N(v) - N(u)$. Recall the auxiliary vertices a_B, a'_B for B, so that a_B sees u, v, and a'_B and misses x and y, while a'_B sees x and y and misses u and v. If xy is an edge in G, then a_B, u, v, x, y, a'_B is a $\overline{C}_6$. If xy is not an edge in G, then a_B, u, v, x, y is a bull. So (iv) is proved.

3.1. Proof of the transitive box partition lemma. Given a box partition, any edge whose endpoints are in different boxes will be called a *vertical edge*. (Necessarily, for any such edge, one endpoint is in an odd box and the other is in an even box.) The other edges will be called *horizontal*; i.e., a horizontal edge is any edge whose two endpoints are in the same box. Recall from (iii) that each box B is P_4-free, and recall that every P_4-free graph admits a transitive orientation [11]. Let $\mathcal{L}(B)$ be a transitive orientation for each box B. All edges xy of G are oriented according to the following rules:

- *Rule* V0. If an edge is vertical, orient it from its even extremity to its odd extremity.
- *Rule* H1. If x, y are in an even (resp., odd) box B and x has strictly more neighbors than y in $M(B)$, then orient xy from x to y (resp., from y to x).
- *Rule* H2. If x, y are in an even (resp., odd) box B and have the same neighborhood in $M(B)$, and if there exists a P_4 $yxvu$ with $u \in M(B)$ and $v \in B$, then orient the edge yx from y to x (resp., from x to y).
- *Rule* H3. If x, y are in a box B and do not satisfy the hypotheses of Rules H1 and H2, then orient xy according to $\mathcal{L}(B)$.

After these rules are applied, every edge of G has received an orientation. We claim that this is a transitive orientation of G. To certify this claim, we have to check that these combined rules are consistent (i.e., noncontradictory) and that they produce no P_3 xyz with orientation $\vec{xy}$ and $\vec{yz}$ and no circuit. Note that a result of Ghouila-Houri [10] shows that if a graph admits an orientation with no directed P_3, then it admits an acyclic transitive orientation.

CLAIM 1. *The rules are consistent.*

Proof. We need only prove that no edge must be oriented by the rules in two opposite ways. Clearly, the vertical edges are oriented consistently. Since Rules H1, H2, and H3 apply to edges of different types, they cannot contradict each other. Rule H1 cannot orient an edge in two opposite ways, by property (iv) of the box partition. Clearly Rule H3 also cannot orient an edge in two opposite ways. So the only case of inconsistency would be the following: some horizontal edge xy ($x, y \in$ box B) must be oriented in one way because there is a P_4 $uvxy$ with $u \in M(B)$ and $v \in B$ (Rule H2) and must also be oriented in the opposite way because there is a P_4 $ztyx$ with $z \in M(B)$ and $t \in B$, and x and y have the same neighbors in $M(B)$. Clearly $v \neq t$ (but $u = z$ is possible). Let a'_B be the auxiliary vertex of B given by property (ii) of the box partition; so a'_B sees u and z and misses all of v, x, y, t. In addition, v must see t or else $vxyt$ is a P_4 in B. Now, either u sees t or z sees v, or else property (iv) is contradicted for v, t. By symmetry we may assume that u sees t, but then v, t, u, a'_B, x is a bull. So Claim 1 is proved. ∎

Claim 2. *The rules produce no P_3 xyz with orientation $\vec{xy}$ and $\vec{yz}$.*

Proof. Suppose the contrary. Rules V0 and H1 imply easily that the vertices x, y, z cannot be in different boxes. So, and by symmetry, we may assume that they lie in one odd box B. Note that one of the edges xy, yz must have been oriented by Rule H1 or by Rule H2.

Case 1. The edge xy was oriented from x to y by Rule H1. This hypothesis means that there exists a vertex u in $M(B) \cap N(y) - N(x)$. If u misses z, then yz should be oriented by Rule H1 from z to y, which is a contradiction; so u sees z. Now x, y, z, u, a'_B is a bull, which is a contradiction.

So xy is not oriented by Rule H1, and then x and y have the same neighborhood in $M(B)$.

Case 2. The edge xy was oriented from x to y by Rule H2. This means that x and y have the same neighborhood in $M(B)$ and that there exists a P_4 $uvxy$ with $u \in M(B)$ and $v \in B$. Because B has no P_4, we have that v sees z. In addition, if u sees z, then u, v, z, y, a'_B is a bull. Moreover, y and z have the same neighborhood in $M(B)$. For, if y has more neighbors than z in $M(B)$, then by Rule H1, we have yz oriented from z to y. If there exists $w \in M(B) \cap N(z) - N(y)$, then w misses x also and Rule H2 orients xy from y to x, which is a contradiction. Thus we can apply Rule H2 to the P_4 $uvzy$ which forces yz to be oriented from z to y, which is a contradiction.

Case 3. The edge yz was oriented from y to z by Rule H1. This hypothesis means that there exists a vertex u in $M(B) \cap N(z) - N(y)$. Recall that x and y have the same neighborhood in $M(B)$. Thus we can apply Rule H2 to the P_4 $uzyx$ which forces xy to be oriented from y to x, which is a contradiction.

Case 4. The edge yz was oriented from y to z by Rule H2. This means that there exists a P_4 $uvyz$ with $u \in M(B)$ and that y and z have the same neighborhood in $M(B)$. Recall that x and y also have the same neighborhood in $M(B)$. Vertex x misses v or else u, v, x, y, z is a bull. Thus we can apply Rule H2 to the P_4 $uvyx$ which forces xy to be oriented from y to x, which is a contradiction.

In all cases a contradiction arises; so Claim 2 is proved. ∎

Claim 3. *The rules produce no circuit.*

Proof. By Rule V0, a circuit may occur only inside a box. Without loss of generality, let us assume that an odd box B contains a circuit $C = c_1 \cdots c_r$. Observe that if an edge xy in B is oriented from x to y, then y has at least as many neighbors as x in $M(B)$ because of Rule H1. Therefore, if somewhere along the circuit two consecutive vertices c_i, c_{i+1} satisfy $N(c_i) \cap M(B) \subset N(c_{i+1}) \cap M(B)$ (where $\subset$ denotes strict inclusion), then necessarily elsewhere on the cycle some two consecutive vertices c_j, c_{j+1} must satisfy $N(c_{j+1}) \cap M(B) \subset N(c_j) \cap M(B)$. But then this inclusion contradicts the fact that the edge $c_j c_{j+1}$ is oriented from c_j to c_{j+1}. So, all vertices along C have the same neighborhood in $M(B)$. Moreover, since $\mathcal{L}(B)$ has no circuit, at least one edge of C must have been oriented by Rule H2. So let us assume that there is a P_4 uvc_1c_2 with $u \in M(B)$ and $v \in B$. Since all the vertices of C have the same neighborhood in $M(B)$, in particular, they all miss u, and v is not one of the c_i's. Let j be the last subscript such that v misses c_j ($j \geq 2$). Then $uvc_{j+1}c_j$ is a P_4 implying that the edge c_jc_{j+1} is oriented from c_{j+1} to c_j, which is a contradiction. So Claim 3 is proved. ∎

Now the proof of Lemma 2.2 is complete.

4. More about the box partition. Everywhere in this section we reserve the letter G for a prime bull-free Berge graph that contains a hole. We let k denote the length of a shortest even hole in G. So there are k nonempty sets $V_1, \dots, V_k$ such that every vertex in V_i sees every vertex in V_{i+1} (modulo k) and there is no other edge between two V_i's. By [7] G admits a box partition built from the V_i's. We will need to use some properties and notation from [7] concerning this box partition. In particular, the boxes of this partition are classified as either "central" or "peripheral" with the following properties that will be used here:

(a) If $k \geq 8$, every central box is a homogeneous set. (To see this, recall from [7] that when $k \geq 8$ the central boxes are the connected components of the k sets $V_1, \dots, V_k$. If $B \subseteq V_1$, then the proof of [7, Lemma 3] gives $M(B) \subseteq A_2$, where A_2 is the set of vertices that are adjacent to all of $V_1 \cup V_3 \cup \dots \cup V_{k-1}$; hence every vertex adjacent to B is adjacent to all of B, and B is a homogeneous set.)

(b) If $k = 6$, there are eight sets $D_1, \dots, D_6, A_1, A_2$ such that the central boxes are exactly the connected components of these eights subgraphs. The sets $D_1, \dots, D_6$ play symmetrical roles; the sets A_1, A_2 play symmetrical roles. Moreover, if B is a box in D_1 or in A_1, then $M(B) \subseteq D_4 \cup A_2$. There are vertices v_2, v_6 that see all of $D_1 \cup A_1$ and none of $D_4 \cup A_2$; there are vertices v_3, v_5 that see all of $D_4 \cup A_2$ and none of $D_1 \cup A_1$; v_2v_3 and v_5v_6 are the only edges between v_2, v_3, v_5, v_6.

(c) [7, Lemma 4, Property (v)] In a peripheral box B any two adjacent vertices have comparable neighborhoods in $M(B)$.

LEMMA 4.1. *The graph G contains an antihole if and only if it contains a $\overline{C}_6$.*

Proof. The "if" part is trivial. Conversely, suppose that G contains no $\overline{C}_6$. Then Lemma 2.2 implies that G is transitively orientable, and hence it contains no antihole. ∎

When a C_4 (a "square") is denoted $uvxy$ it is understood that ux and vy are the two nonadjacent pairs.

DEFINITION 4.2 (blocking square). *We say that a square $uvxy$ is blocking if u, v belong to one box B and x, y belong to another box B'. The edges uv and xy are called the blocking edges of the square. Likewise any edge uv with both endpoints in one box is called blocking whenever it is one of the two blocking edges of a blocking square.*

Remark 2. In Definition 4.2, clearly one of B, B' is an even box and the other is an odd box. Clearly too, we have $\{x, y\} \subseteq M(B)$ and $\{u, v\} \subseteq M(B')$.

LEMMA 4.3. *The graph G contains a $\overline{C}_6$ if and only if G contains a blocking square.*

Proof. First suppose that G contains a blocking square $uvxy$ with the notation as in Definition 4.2. Let a_B and a'_B be the auxiliary vertices for B. Then a_B, a'_B, u, v, x, y induce a $\overline{C}_6$.

Conversely, suppose that six vertices $u_1, u_2, \dots, u_6$ form in G a $\overline{C}_6$, such that the nonadjacent pairs are u_iu_{i+1} (subscripts here are understood modulo 6). In the triangle u_1, u_3, u_5 at least two vertices are on the same side of the box partition; say, u_1 and u_3 are in one even box B. If both u_4, u_6 are in an odd box, then $u_1u_3u_6u_4$ is a blocking square. So let us assume without loss of generality that u_4 is in an even box and hence in B. Then u_2 is in an odd box or else $u_3u_1u_4u_2$ would be a P_4 in B. If u_5 is on the odd side, then $u_1u_4u_2u_5$ is a blocking square. So let us assume that u_5 is on the even side and hence in B. Then u_6 is on the odd side or else $u_5u_3u_6u_4$ would be a P_4 in B. But now $u_3u_5u_2u_6$ is a blocking square. ∎

LEMMA 4.4. *An edge uv in a box B is a blocking edge if and only if the vertices u, v have incomparable neighborhoods in $M(B)$.*

Proof. The "only if" part of the lemma is trivial. Conversely, suppose that u, v have incomparable neighborhoods in $M(B)$; i.e., there exist a vertex $x \in M(B) \cap N(v) - N(u)$ and a vertex $y \in M(B) \cap N(u) - N(v)$. Recall that the auxiliary vertex a_B sees both u, v and misses both x, y. Then x must see y or else a_B, u, v, x, y would be a bull. Now $uvxy$ is a blocking square and uv is a blocking edge. ∎

At this point it is useful to recall the graph called H_0 in [3] and featured in Figure 2.

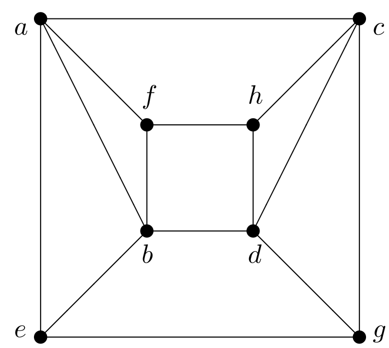

FIG. 2. *The graph H_0.*

LEMMA 4.5. *If G contains an antihole, then G contains an H_0.*

Proof. By the preceding lemmas we may assume that G admits a blocking square $uvxy$, with blocking edges uv in a box B and xy in a box B'. So the vertices u, v have incomparable neighborhoods in $M(B)$. By [7, Lemma 4, Property (v)] as recalled above, B cannot be a peripheral box. So B is a central box. If $k \geq 8$, then item (a) above implies that B is a proper homogeneous set, which is impossible because u, v have incomparable neighborhoods in $M(B)$. So we have $k = 6$, and we may assume, without loss of generality, that $B \subseteq D_1 \cup A_1$, with the notation of item (b). Now we have

$$u, v \in B \subseteq D_1 \cup A_1, \qquad x, y \in B' \subseteq D_4 \cup A_2.$$

Using the vertices v_2, v_3, v_5, v_6 whose properties are recalled in (b) above, we note that vertex v_2 sees all of $\{u, v, v_3\}$, vertex v_6 sees all of $\{u, v, v_5\}$, vertex v_3 sees all of $\{x, y, v_2\}$, vertex v_5 sees all of $\{x, y, v_6\}$, and there are no other edges between the vertices $u, v, x, y, v_2, v_3, v_5, v_6$. Hence these eight vertices induce an H_0. ∎

The following result will be useful. Recall that, given a homogeneous pair Q_1, Q_2 in a graph G, we denote by T_i, $i = 1, 2$, the set of vertices in $G \setminus (Q_1 \cup Q_2)$ that see all of Q_i and miss all of Q_{3-i}, by T the set of vertices in $G \setminus (Q_1 \cup Q_2)$ that see all of $Q_1 \cup Q_2$, and by Z the set of vertices in $G \setminus (Q_1 \cup Q_2)$ that see none of $Q_1 \cup Q_2$.

THEOREM 4.6 (see [3]). *Let G be a bull-free graph that contains an H_0 (with the notation as in Figure 2). Then the following hold.*

(i) *G contains a homogeneous pair Q_1, Q_2 such that $a, b \in Q_1$, $c, d \in Q_2$, $e, f \in T_1$, $g, h \in T_2$, and $G[Q_1]$ and $G[Q_2]$ are connected.*

(ii) *If G is connected and prime, then $Z = \emptyset$.*

Proof. Part (i) of the theorem is proved in [3, Theorem 2]; it consists in a polynomial-time algorithm that builds the homogeneous pair Q_1, Q_2 from a given H_0.

To prove part (ii) suppose on the contrary that $Z \neq \emptyset$. Since G is connected, there exists an edge zt with $z \in Z$ and $t \in T \cup T_1 \cup T_2$. If $t \in T_1$, then z, t, a, b, c induce a bull, which is a contradiction. So we may assume that $t \in T$, and z misses e since $e \in T_1$. Then t sees e, or else z, t, e, a, c would induce a bull. But then z, t, e, a, c, d induce a broom, which is a contradiction to Lemma 3.2. ∎

5. Proof of Theorem 2.3. Let G be a prime bull-free Berge graph that contains a hole and an antihole. Recall that we want to prove that (I) the graph G contains an interesting homogeneous pair Q_1, Q_2; (II) if H, Q are the two graphs obtained by decomposing G along an interesting homogeneous pair, then both H and Q are bull-free Berge graphs; and (III) it is possible to build a solution of the maximum weighted clique problem on G from a solution of the same problem on H and Q with appropriately defined vertex-weights.

To prove (I), we need only apply Lemma 4.5 and Theorem 4.6 above.

Now let us prove (II). Let $abcd$ be a square with edge ab in Q_1 and edge cd in Q_2. Here again T_i (resp., T) is the set of vertices of $G \setminus (Q_1 \cup Q_2)$ that see all of Q_i and none of Q_{3-i} (resp., all of $Q_1 \cup Q_2$), and $Z = V \setminus (Q_1 \cup Q_2 \cup T_1 \cup T_2 \cup T)$; i.e., no vertex of Z sees any of $Q_1 \cup Q_2$.

Recall that the graph H is obtained from $G \setminus (Q_1 \cup Q_2)$ by adding vertices u_1, u_2, s_1, s_2, edges u_1s_1, u_2s_2, u_1s_2, u_2s_1, s_1s_2, and edges tu_i, ts_i for each i and each vertex $t \in T_i \cup T$.

LEMMA 5.1. *H is perfect and bull-free.*

Proof. Call G^* the subgraph of G induced by $V \setminus ((Q_1 \setminus \{a, b\}) \cup (Q_2 \setminus \{c, d\}))$. Observe that $H \setminus s_1s_2$ is isomorphic to G^*.

First we prove that H is perfect. Consider any induced subgraph H' of H. If H' contains at most one of u_1, s_1 and at most one of u_2, s_2, then H' is isomorphic to one of the subgraphs $G^* \setminus \{a, c\}$, $G^* \setminus \{a, d\}$, so H' is perfect. Suppose now by symmetry that H' contains both u_1 and s_1. Note that s_1 dominates u_1 in H (i.e., $N_H(u_1) \subset N_H(s_1) \cap \{s_1\}$), and thus also in H'. It is well known (see, e.g., [11]) that a minimally imperfect graph cannot contain a pair of vertices such that one dominates the other. So all induced subgraphs of H are perfect, including H itself.

Now suppose that H contains a bull B. It is easy to see that any induced subgraph of H that contains none of the two triangles formed by u_1, s_1, s_2 and u_2, s_1, s_2 is contained in one of the subgraphs $G^* \setminus \{a, c\}$, $G^* \setminus \{a, d\}$ and thus cannot be a bull. So we may assume by symmetry that B contains the triangle u_1, s_1, s_2. Now B must have a vertex adjacent to exactly one of u_1, s_1 and not adjacent to s_2. But H contains no such vertex since u_2 is the only vertex adjacent to exactly one of u_1, s_1. This completes the proof of the lemma. ∎

Since Q is the subgraph of G induced by $Q_1 \cup Q_2$, the next claim is obvious.

CLAIM 4. *Q is perfect and bull-free.* ∎

We now prove part (III) of Theorem 2.3. Let us denote by $w(x)$ the weight of a vertex x in G. Define weights for vertices in H as follows. Denote by $\omega(X)$ the maximum weight of a clique in X, and set

$$\begin{aligned} w_H(u_1) = w_H(u_2) &= \omega(Q_1) + \omega(Q_2) - \omega(Q_1 \cup Q_2), \\ w_H(s_1) &= \omega(Q_1 \cup Q_2) - \omega(Q_2), \\ w_H(s_2) &= \omega(Q_1 \cup Q_2) - \omega(Q_1), \\ w_H(x) &= w(x) \qquad \forall x \in G \setminus (Q_1 \cup Q_2). \end{aligned}$$

Say that a set X of vertices in H is of type 0 if $X \cap \{u_1, s_1, u_2, s_2\} = \emptyset$, of type 1 if $X \cap \{u_1, s_1\} \neq \emptyset$ and $X \cap \{u_2, s_2\} = \emptyset$, of type 2 if $X \cap \{u_1, s_1\} = \emptyset$ and $X \cap \{u_2, s_2\} \neq \emptyset$, and of type 3 if $X \cap \{u_1, s_1\} \neq \emptyset$ and $X \cap \{u_2, s_2\} \neq \emptyset$.

Let q be the maximum weight of a clique in H with respect to the weighting w_H, and let C_H be a clique of weight q. We can transform C_H into a clique C_G of weight q in G as follows. If C_H is of type 0, set $C_G = C_H$. If C_H is of type $i \in \{1,2\}$, let C_G be the union of $C_H \setminus (Q_1 \cup Q_2)$ and of a clique of size $\omega(Q_i)$ in Q_i. If C_H is of type 3, let C_G be the union of $C_H \setminus (Q_1 \cup Q_2)$ and of a clique of size $\omega(Q_1 \cup Q_2)$ in $Q_1 \cup Q_2$.

LEMMA 5.2. *We have $\omega(G) = q$ and C_G is a maximum weighted clique of G.*

Proof. We need only exhibit a q-weighted coloring of G: that will prove both that the clique C_G defined above for G is maximum and that this coloring has minimum weight. The proof of this lemma is essentially the weighted version of the proof of [3, The Homogeneous Pair Lemma].

Recall that $q = \omega(H)$. So there exists a weighted coloring of H of total weight q, that is, a collection of stable sets $S_1^H, \dots, S_t^H$ of H with corresponding weights $W(S_1^H), \dots, W(S_t^H)$, such that

$$\sum\{W(S_i^H) \mid S_i^H \ni x\} = w_H(x) \qquad (\forall x \in H)$$

and $W(S_1^H) + \cdots + W(S_t^H) = q$. Split the subscripts $1, 2, \dots, t$ into sets I_0, I_1, I_2, I_3 by writing $j \in I_i$ if and only if S_j^H is of type i. Thus

$$\begin{aligned} w_H(u_i) &\le \sum\{W(S_j^H) \mid j \in I_i \cup I_3\}, \\ w_H(s_i) &\le \sum\{W(S_j^H) \mid j \in I_i\}. \end{aligned}$$

In addition, since u_i and s_i are adjacent,

$$w_H(u_i) + w_H(s_i) \le \sum\{W(S_j^H) \mid j \in I_i \cup I_3\}.$$

Define a graph F by adding to the subgraph $Q = G[Q_1 \cup Q_2]$ adjacent vertices x_1, x_2 and edges $x_i y$ for all vertices y in Q_i ($i = 1, 2$). Note that F is isomorphic to the subgraph of G induced by $Q_1 \cup Q_2 \cup \{t_1, t_2\}$, where $t_1 \in T_1$, $t_2 \in T_2$, and $t_1 t_2$ is an edge of G; such vertices exist because Q_1, Q_2 is an interesting homogeneous pair. So F is a perfect graph. Define a weight function W_F on the vertices of F as follows:

$$\begin{aligned} w_F(x_1) &= \sum\{W(S_j^H) \mid j \in I_2\}, \\ w_F(x_2) &= \sum\{W(S_j^H) \mid j \in I_1\}, \\ w_F(y) &= w(y) \qquad (\forall y \in Q_1 \cup Q_2). \end{aligned}$$

We have

$$\begin{aligned} w_F(x_1) + \omega(Q_1) &= w_F(x_1) + w_H(u_1) + w_H(s_1) \\ &\le \sum\{W(S_j^H) \mid j \in I_1 \cup I_2 \cup I_3\}, \end{aligned}$$

and similarly

$$\begin{aligned} w_F(x_2) + \omega(Q_2) &= w_F(x_2) + w_H(u_2) + w_H(s_2) \\ &\le \sum\{W(S_j^H) \mid j \in I_1 \cup I_2 \cup I_3\}. \end{aligned}$$

In addition,

$$\begin{aligned} \omega(Q_1 \cup Q_2) &= w_H(u_1) + w_H(s_1) + w_H(s_2) \\ &\le \sum\{W(S_j^H) \mid j \in I_1 \cup I_2 \cup I_3\}. \end{aligned}$$

Hence each clique C_F has weight at most $q_F = \sum\{W(S_j^H) \mid j \in I_1 \cup I_2 \cup I_3\}$. Since F is perfect, there exist a family of stable sets $S_1^F, \dots, S_r^F$ of F and weights $W(S_1^F), \dots, W(S_r^F)$ such that

$$W(S_1^F) + \cdots + W(S_r^F) \leq q_F$$

and

$$\sum\{W(S_j^F) \mid x \in S_j^F\} = w_F(x) \qquad (\forall x \in F).$$

Since x_1 and x_2 are adjacent, no S_j^F contains both x_1, x_2. The definition of $w_F(x_1)$ implies

$$\sum\{W(S_j^F) \mid x_1 \in S_j^F\} = \sum\{W(S_j^H) \mid j \in I_2\},$$

and similarly for x_2.

Now we build a family of stable sets of G by "merging" the families of stable sets of H and of F defined above. This is done as follows: First we merge the family $\{S_i^H \mid i \in I_1\}$ with the family of those stable sets S_j^F that cover x_2. Note that the total weight is the same for both families, by the definition of $w_F(x_2)$, though the individual weights may be different. Also, each set $(S_j^F \cap (Q_1 \cup Q_2)) \cup (S_i^H \setminus (Q_1 \cup Q_2))$ is a stable set, because the choice of j, i is such that S_j^F covers x_2 and $i \in I_1$.

Merging procedure. Take the heaviest set S of the two families (say, the first family), then take the heaviest set T of the second family, and merge them. That is, make the set $S \cup T \setminus \{u_1, s_1, x_2\}$; remove S and T from their respective families; if the weight α of S is strictly larger than the weight β of T, put a copy of S in the first family with weight $\alpha - \beta$; repeat with the remaining families until they are emptied out. Clearly, at the end of each step of the merging subroutine at least one of the two families has one less element, so the merging procedure produces a finite family of stable sets of G (more precisely, the total number of steps, and thus of merged sets that are created, is at most the total size of the two families).

Likewise, we merge the family $\{S_i^H \mid i \in I_2\}$ with the family of stable sets S_j^F covering x_1. Note that these two families have the same total weight, by the definition of $w_F(x_1)$.

Likewise, we merge the family $\{S_j^H \mid j \in I_3\}$ with the remaining family of stable sets S_j^F (i.e., those stable sets S_j^F that do not cover any of x_1, x_2).

Finally, the three families of stable sets produced by the mergings above, plus the family $\{S_j^H \mid j \in I_0\}$, form a family $S_1, \dots, S_t$ of stable sets of G with weights $W(S_1), \dots, W(S_t)$, such that

$$\sum_{S_i \ni v_j} W(S_i) = w(v_j) \qquad (\forall v_j). \tag{3}$$

Since the total weight of $S_1, \dots, S_t$ is q, these stable sets form a minimum weighted coloring of G, and this certifies that C_G is a maximum weighted clique of G.

This completes the proof of Lemma 5.2 and of Theorem 2.3. $\square$

6. The algorithm. We can now present the algorithm BFCLIQUE, which, given a bull-free Berge graph $G = (V, E)$ with a weight $w(x)$ on each vertex x, finds in polynomial time a maximum weighted clique of G. Along with the description of the algorithm it is convenient to maintain a decomposition tree T_G associated with G.

Step 1. In a first phase, we test whether G has any nontrivial homogeneous set. Determining the homogeneous sets of a graph is a problem that is essentially solved by the theory of *modular decomposition*; see, in particular, [4, 6, 17], stemming from the seminal work of Gallai [9, 16]. This theory is rich and complex, and we outline only the aspects that will be used here. Say that a homogeneous set S is *strong* if every homogeneous set S' satisfies $S' \subseteq S$ or $S \subseteq S'$ or $S' \cap S = \emptyset$. It is known [9, 16] that the strong homogeneous sets form a nested family, and so there are at most $2n$ of them, including V and every singleton $\{v\}$ $(v \in V)$. One can associate with every graph G a unique rooted tree M_G defined as follows. Let $X_1, \ldots, X_r$ be the (inclusionwise) maximal strong homogeneous sets of G, and let G' be the graph obtained from G by contracting each X_i into one vertex x_i. The root of M_G is G, and the children of node G in M_G are the graphs $G[X_1], \ldots, G[X_r], G'$. For each $i = 1, \ldots, r$, the subtree of M_G rooted at node $G[X_i]$ is the tree $M_{G[X_i]}$ defined recursively. As for G', it follows from the theory of modular decomposition that G' is a clique, or an edgeless graph, or a prime graph, so G' is a leaf of M_G. This tree is called the *modular decomposition tree* of G and can be computed in time linear in the number of edges of G [4, 6, 17].

To obtain a maximum weighted clique for G, we can follow this tree from the bottom up. Assume that we have a maximum weight clique $Q(X_i)$ for each graph $G[X_i]$. We then assign the weight of $Q(X_i)$ to vertex x_i in G'. We then apply the algorithm BFCLIQUE (step 2) on G' and obtain a clique Q' of maximum weight in G'. From Q' we can obtain a clique Q of G by replacing any x_i that lies in Q' by the vertices of $Q(X_i)$. Then Q is a maximum weighted clique of G. (To see this, take a minimum weighted coloring for each of the $G[X_i]$'s, of weight $\omega(G[X_i])$ since $G[X_i]$ is perfect, and for G', of weight $\omega(G')$ since G' is perfect, and merge them in the obvious way; thus a weighted coloring is obtained for G, whose weight is the weight of Q.)

This first step shows that the computation of a maximum clique for G by BFCLIQUE is reduced to calls of BFCLIQUE on at most n graphs (the leaves of the modular decomposition tree). We represent this situation in the associated tree T_G by saying that if G has a nontrivial homogeneous set, then the children of node G in T_G are the leaves of the modular decomposition tree M_G. In that case we say that node G is a modular node of T_G.

Step 2. We are now dealing with a bull-free Berge graph K that is a clique, or an edgeless graph, or a prime graph. We use the algorithm from [7], whose output is one of the following cases.

2.1. K is weakly triangulated. We use the algorithm due to Hayward, Hoàng, and Maffray [14] to produce a maximum weighted clique of K; it is strongly polynomial and its time complexity is $\mathrm{O}(n^4m)$. Since K is not subject to a decomposition, node K is a leaf of T_G.

2.2. K contains an even hole and the algorithm produces a box partition for K, and there is no blocking square with respect to this partition. Lemmas 4.1 and 4.3, Theorem 2.1, and Lemma 2.2 imply that K is transitively orientable. A transitive orientation can be found in linear time using the algorithm in [17]. A maximum weighted clique and a minimum weighted coloring can be found using the algorithm in [15], which is strongly polynomial and whose time complexity is $\mathrm{O}(nm)$. Here too, node K is a leaf of T_G.

2.3. K contains an even hole and the algorithm produces a box partition for K, and there is a blocking square. Then Lemmas 4.1 and 4.5, and Theorem 2.3 imply that we can decompose K into the two graphs H and Q as above. The proof of part (III) of Theorem 2.3 describes how to obtain a solution to our problem on K from a solution of the same problems on each of H, Q. Therefore, H and Q are the two children of node K in the tree T_G. We will say that node K is an H_0-node of T_G.

2.4. K contains no even hole, its complementary graph $\overline{K}$ contains an even hole, and the algorithm produces a box partition for $\overline{K}$. Then K is a leaf of T_G. Lemmas 4.3 and 4.1, Theorem 2.1, and Lemma 2.2 imply that $\overline{K}$ is transitively orientable. A transitive orientation of $\overline{K}$ can be found rapidly using the algorithm in [17]. Finding a maximum weighted clique and a minimum weighted coloring for K is equivalent to finding a maximum weighted stable set and a minimum weighted clique covering for the transitively orientable graph $\overline{K}$; this problem can be solved in strongly polynomial time by the algorithm described in [2]. Here too, node K is a leaf of T_G.

7. Complexity analysis. As noted several times, each step of the algorithm can be done in polynomial time. So, in order to prove polynomiality of the whole algorithm, we need only establish the following lemma.

LEMMA 7.1. *There is a polynomial number of nodes in the tree T_G.*

Proof. Let n and m be the number of vertices and edges in G. There are two types of nonleaf nodes in T_G: modular nodes and H_0-nodes. Let $\beta(G), \beta_1(G), \beta_0(G)$ be, respectively, the number of nodes, of modular nodes, and of H_0-nodes in T_G. Note that each node of T_G has no more vertices than its parent. (The "H" child of an H_0-node K may have exactly as many vertices as K; this happens if its sibling the "Q" child of K has exactly four vertices.) Thus every node of T_G has at most n vertices. So each node has at most n children, and $\beta(G) \leq n(\beta_0(G)+\beta_1(G))$. The principle of modular decomposition implies that the parent of a modular node is an H_0-node. Since each H_0-node has exactly two children, we see that $\beta_1(G) \leq 2\beta_0(G) + 1$. Therefore, $\beta(G) \leq n(3\beta_0(G) + 1)$, and we need only prove that $\beta_0(G)$ is a polynomial of n. Our counting argument now focuses on the subgraphs (of nodes of T_G) that induce a $2K_2$, i.e., a graph on four vertices with two nonincident edges. We want to see how the number of $2K_2$'s evolves along T_G.

First, suppose that K is a modular node of T_G. Then it is a routine matter to check that the total number of $2K_2$'s that are induced in its children in M_G is not larger than the number of $2K_2$'s induced in K, and thus that the same holds with respect to the children of K in T_G.

Second, suppose that K is an H_0-node of G, decomposed along an interesting homogeneous pair Q_1, Q_2 (with the notation T_1, T_2, T, Z as usual), and call H, Q the two children of K in the tree. Let us prove that *in total H and Q have strictly fewer $2K_2$'s than K*. For this purpose let us define a one-to-one mapping f that maps every subgraph D that induces a $2K_2$ in H or Q to a subgraph $f(D)$ that induces a $2K_2$ in K. If D is in Q, then set $f(D) = D$. If D is in H, we observe that D does not have an edge with an endvertex in $\{u_1, s_1\}$ and the other in $\{u_2, s_2\}$, for otherwise the remaining two vertices of D should be in Z, which is a contradiction to (ii) of Theorem 4.6. Therefore, if D is in H, we let $f(D)$ be the $2K_2$ of K obtained from D by replacing u_1, s_1, s_2, u_2 (whichever appear in D), respectively, by a, b, c, d. It is a routine matter to check that f is indeed a one-to-one mapping. Moreover, the $2K_2$ of G induced by b, c, e, h is not the image $f(D)$ of any $2K_2$ D of H or Q. This ensures that H and Q have in total strictly fewer $2K_2$'s than K.

Now let T'_G be the tree obtained from T_G by contracting each node that is not an H_0-node with its parent. The number of nodes of T'_G is $\beta_0(G)$ (if G is an H_0-node) or $\beta_0(G) + 1$ (if G is a modular node). The preceding two paragraphs imply that the total number of $2K_2$'s at a given level of T'_G decreases strictly as the level is farther from the root (viewed as level 0), with the only possible exception of the first level if G is an H_0-node. This implies that the number of nodes in T'_G is bounded by the number of $2K_2$'s in G plus 1, which is $\mathrm{O}(m^2)$. □

With this final claim, we obtain that the total number of recursive calls to the algorithm is at most $\mathrm{O}(m^2)$. It follows that the algorithm is strongly polynomial, with worst-case complexity $O(n^5 m^3)$.

Let us conclude with a remark. The proof of Lemma 5.2 shows how a minimum weighted coloring can be found directly from a minimum weighted coloring of H and of the graph F defined in Lemma 5.2. This method could be the basis for a coloring algorithm that does not involve n^4 calls to the maximum weighted clique algorithm as mentioned in the introduction. However, this method leads to a decomposition algorithm in which a node K that contains an H_0 must be decomposed into three graphs H, Q, F. In that case, note that the vertices of Q also appear in F, and so we cannot guarantee that the total number of nodes of the decomposition tree remains polynomial in the size of the root graph.

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