Contributions to Discrete Mathematics

Volume 19, Number 3, Pages 121–136
ISSN 1715-0868

CANONICAL CUTS OF PATH POWERS

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Abstract. The MaxCut problem aims to find a bipartition of vertices in a given graph such that the number of edges with one end vertex in each part is maximum among all bipartitions. NP-hardness when restricted to interval graphs has been recently announced. Surprisingly, all previously published attempts at polynomial-time algorithms for unit interval graphs turned out to be wrong, which justifies the search for subclasses where MaxCut can be handled. We introduce canonical cuts whose pattern allows an easy computation of the cut size for the power of paths P_n^k. Using canonical cuts, we calculate the structure and the size of maximum cuts for $k \leq 5$ and for $n \leq \frac{2}{3}(2k+2)$. We prove that the known size for a maximum cut for reduced co-bipartite chain graphs can be achieved by a canonical cut. We perform computational experiments on each P_n^k graph with $1 \leq k \leq n \leq 43$ and show that most of them allow a canonical cut that is maximum. We display a table with the found cases where there is no canonical cut which is a maximum cut. In these graphs, the difference between the maximum cut and some canonical is at most 3 units. This indicates canonical cuts as a good approach to tackle the maximum cut on P_n^k graphs.

1. Introduction

The MaxCut problem aims to find in a given graph a bipartition of its vertices such that the number of edges with one end vertex in each part is maximum among all bipartitions. Although positive weights on the edges may be additionally considered, we focus on the unweighted simpler case, also known as the simple MaxCut problem. MaxCut is a well-studied problem listed as [ND16] in [11], proved to be NP-complete [10] even when restricted to cubic graphs [3], split graphs [4], co-bipartite graphs [4], unit disk graphs [9], and total graphs [13], while proved to be polynomial-time solvable when restricted to planar graphs [14], line graphs [13], graphs not contractible to K_5 [2], co-bipartite chain graphs [7], circulant graphs [16],

Received by the editors April 7, 2021, and in revised form July 16, 2022.
2000 *Mathematics Subject Classification.* 05C70, 05C85, 68R10.
Key words and phrases. MaxCut, power of paths, chain graphs, unit interval graphs.

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graphs with bounded treewidth [4], and graphs that are both split and unit interval [6]. NP-hardness when restricted to interval graphs has been recently announced [1]. All previously published attempts to polynomial-time algorithms for unit interval graphs [5, 8] turned out to be wrong [6, 15].

We consider the subclass of interval graphs formed by powers of paths P_n^k. The powers of paths are actually unit interval graphs since they admit an interval graph representation with all intervals of the same size. We introduce the b-canonical cuts which can be roughly described as the sequential division of the vertices of P_n^k at blocks of length b. The regular pattern of canonical cuts allows an easy computation of their size. We remark that in order to have a regular pattern, it is important to consider the MaxCut problem for the unweighted case, where all edges have equal weight 1. We propose and study the following question:

Question. *For which values of k and n does there exists $b = b(k, n)$ such that the b-canonical cut of P_n^k is a maximum cut?*

We partially answered this question for small k and arbitrary n; and for arbitrary k and $n \leq \frac{2}{3}(2k+2)$. We have also verified it for an arbitrary k and $n \in \{2k + 1, 2k + 2\}$, which completes and improves the result given in [7]. Notice that the existence of a b-canonical cut that is maximum implies that MaxCut is polynomial-time solvable in the respective P_n^k.

A challenging and meaningful combinatorial problem is to determine for which values of b the b-canonical cut of P_n^k is a maximum cut. For $k \leq 5$ we have established such b, and we have observed it depends only on k (it is independent of n), that is we answered the question for $k \leq 5$ and obtained the characterization of some cuts that are maximum for these classes of power of paths. On the other hand, for $k = 6$, we proved that in order to obtain a maximum cut of P_n^6, it is not possible to choose the same b for all n, however we have verified an asymptotic behavior. That is, we have proved that for n big enough, the size of the 5-canonical cut of P_n^6 is larger than the size of any other canonical cut.

Section 2 presents the required notation and technical results which give the tools used to evaluate the canonical and maximum cut sizes, in Section 2 for $n \leq \frac{2}{3}(2k + 2)$ and in Section 3 for $k \leq 5$. In Section 4, we study the particular behavior of the case $k = 6$. In Section 5, we consider the cases $n = 2k + 1$ and $n = 2k + 2$, and prove that the known size for a maximum cut for reduced co-bipartite chain graphs [7] can be achieved by a canonical cut. In Section 6, we present computational evidence that for some values of k and n the maximum cut is not canonical. Our concluding remarks are in Section 6.

2. Notation and technical results

Let G be a graph with vertex set and edge set $V(G)$ and $E(G)$, respectively. For S and S' disjoint subsets of $V(G)$, we let (S', S) denote the set of edges of G with one end vertex in S and the other in S'. Clearly, $(S, S') = (S', S)$.

When S' is the complement of S (i.e. $S' = \overline{S} = V(G) \setminus S$), we say that (S, S') is a *cut* of G. The cardinality of $(S, \overline{S})$ is the *size* of the cut. A cut is said to be *maximum* if it has maximum size among all the cuts of G. The size of a maximum cut of G is denoted by $mc(G)$. Bipartite graphs B satisfy $mc(B) = |E(B)|$, and complete graphs K_n satisfy $mc(K_n) = \lfloor n/2 \rfloor \lceil n/2 \rceil$.

Let P_n be the chordless path $v_1, v_2, \ldots, v_n$. For a positive integer k, the kth *power* of P_n is the simple graph P_n^k obtained from P_n by adding an edge between every pair of vertices at distance at most k. Formally, $V(P_n^k) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n^k) = \{v_i v_j | 1 \leq i, j \leq n, 0 < |i - j| \leq k\}$. Henceforth, we will assume $n \geq 2$.

A *co-bipartite chain* graph is a co-bipartite graph in which the neighborhoods of the vertices in each clique can be linearly ordered with respect to inclusion. Two adjacent vertices $u, v \in V(G)$ are *twins* if $N(u) \setminus v = N(v) \setminus u$. A graph is *reduced* if it does not contain twin vertices. A *unit interval* graph (also known as an indifference or proper interval graph) admits a set of unitary intervals $\mathcal{S}$ on the real line and a bijection ϕ from $V(G)$ to $\mathcal{S}$ such that vertices u, v are adjacent if and only if $\phi(u) \cap \phi(v) \neq \emptyset$. A *split* graph is a graph whose vertex set can be partitioned into a stable set and a clique.

Remark: The following assertions hold trivially.

- If $k = 1$, then $P_n^k = P_n$.
- If $n \leq k + 1$, then P_n^k is the complete graph K_n.
- If $n = k + 2$, then P_n^k is the graph obtained from K_n by removing one edge, and has exactly two maximal cliques each of size $n - 1$.
- If $n = k + 3$, then P_n^k is the graph obtained from K_n by removing three edges such that the resulting graph has exactly three maximal cliques each of size $n - 2$.
- P_n^k is a graph that is both split and unit interval if and only if $n \leq k + 3$.
- P_n^k has no twin vertices (reduced) if and only if $n \geq 2k + 1$.
- P_n^k is co-bipartite chain if and only if $n \leq 2k + 2$.
- P_n^k is reduced co-bipartite chain if and only if $n = 2k+1$ or $n = 2k+2$.

In order to simplify the statement of our results, we will describe the cuts of P_n^k by means of an ordered sequence of positive integers $b_1, b_2, \ldots, b_\ell$ with $\ell \geq 1$ and $b_1 + b_2 + \cdots + b_\ell = n$. Such a sequence will represent the cut $(S, \overline{S})$ where the vertices of P_n^k are ordered as in P_n, and the first b_1 vertices of P_n^k are in S, the next b_2 vertices are in $\overline{S}$, the following b_3 vertices are in S, and so on until the n vertices are distributed alternately between the sets S and $\overline{S}$. For instance, the sequence $2, 3, 3, 3, 2$ represents the cut $C = (S, \overline{S})$ of P_{13}^4 defined by $S = \{v_1, v_2, v_6, v_7, v_8, v_{12}, v_{13}\}$. See Figure 1.

For positive integers i, s, and t such that $i+s+t \leq n$, consider the following two subsets of consecutive vertices of P_n^k: $A_s = \{v_{i+1}, v_{i+2}, \ldots, v_{i+s}\}$ and $A_t = \{v_{i+s+1}, v_{i+s+2}, \ldots, v_{i+s+t}\}$. Clearly, $|(A_s, A_t)|$ does not depend on the initial vertex v_{i+1}, it depends only on the values k, s, and t. Therefore, we define $a_{k,s,t}$ to be $|(A_s, A_t)|$ for any sequence of $s + t \leq n$ consecutive vertices of P_n^k. To simplify the notation, whenever the context is clear, we will write $a_{s,t}$ instead of $a_{k,s,t}$. The following two lemmas can be easily proved.

Lemma 2.2. *Let s and t be positive integers, and $n \geq s+t$.*

(1) $a_{k,s,t} = a_{k,t,s}$.
(2) *If $s+t \leq k+1$, then $a_{k,s,t} = s.t$.*
(3) *If $s+t > k+1$ and $s,t \leq k$, then $a_{k,s,t} = s.(k-s) + \frac{1}{2}(s+t-k)(s-t+k+1)$.*

Lemma 2.3. *Let $b_1, b_2, \ldots, b_\ell$ be a cut C of P_n^k. If $b_i + b_{i+1} \geq k$ for $2 \leq i \leq \ell-2$, then $|C| = \sum_{1\leq i\leq \ell-1} a_{b_i,b_{i+1}}$.*

Consider again the example $2,3,3,3,2$ on P_{13}^4 depicted in Figure 1; by Lemma 2.3, we have $|C| = a_{2,3} + a_{3,3} + a_{3,3} + a_{3,2} = 2a_{2,3} + 2a_{3,3}$. By Lemma 2.2, $a_{2,3} = 2.3 = 6$ and $a_{3,3} = 3(4-3) + \frac{1}{2}(3+3-4)(3-3+4+1) = 8$. Therefore, $|C| = 2.6 + 2.8 = 28$.

Definition 2.4. *For a given positive integer $b \leq n$, let $q = \lfloor n/b \rfloor$ and $r = n - q.b$. The b-canonical cut of P_n^k is defined as follows:*

- *if $r \leq \lceil b/2 \rceil$, then the b-canonical cut is $b_1, b_2, \ldots, b_q, b_{q+1}$ where $b_1 = \lfloor (b+r)/2 \rfloor$, $b_2 = \cdots = b_q = b$, and $b_{q+1} = \lceil (b+r)/2 \rceil$;*
- *if $r > \lceil b/2 \rceil$, then the b-canonical cut is $b_1, b_2, \ldots, b_q, b_{q+1}, b_{q+2}$ where $b_1 = r - \lfloor b/2 \rfloor$, $b_2 = \cdots = b_{q+1} = b$, and $b_{q+2} = \lfloor b/2 \rfloor$.*

We say that b is the size of the internal blocks of the cut.

For P_{13}^4, the cut $2,3,3,3,2$ is the 3-canonical cut. For P_{12}^5, the cut $2,4,4,2$ is the 4-canonical cut. For P_{52}^{47}, the cut $2,9,9,9,9,9,5$ is the 9-canonical cut. For P_{202}^{80}, the cut $13,54,54,54,27$ is the 54-canonical cut.

We ask for which values of k and n there exists b such that the b-canonical cut of P_n^k is a maximum cut. Clearly, the 1-canonical cut of P_n^1 is maximum. Lemmas 2.5 and 2.6 answer this question for $n \leq \frac{2}{3}(2k+2)$.

Lemma 2.5. *If $n \leq k+1$, then the $\lfloor n/2 \rfloor$-canonical cut and the $\lceil n/2 \rceil$-canonical cut of P_n^k are both maximum; and $mc(P_n^k) = \lfloor n/2 \rfloor \lceil n/2 \rceil$. Furthermore, if $n \leq k+1$ and $b \geq 2n/3$, then the b-canonical cut of P_n^k is maximum.*

Proof. First notice that if $n \leq k+1$, then P_n^k is a complete graph, therefore $mc(P_n^k) = \lfloor n/2 \rfloor \lceil n/2 \rceil$. If $b = \lfloor n/2 \rfloor$, then $q = \lfloor n/b \rfloor = 2$ and $r \in \{0,1\}$, which implies $r \leq \lceil b/2 \rceil$. Thus, we have that the b-canonical cut is $\lfloor (b+r)/2 \rfloor$, b, $\lceil (b+r)/2 \rceil$ with size $\lfloor n/2 \rfloor \lceil n/2 \rceil$.

If $b = \lceil n/2 \rceil$, we can assume that n is odd, then $q = \lfloor n/b \rfloor = 1$ and $r = n - q.b = n - 1\lceil n/2 \rceil = \lfloor n/2 \rfloor$. If $r \leq \lceil b/2 \rceil$, then the b-canonical cut is $\lfloor (b+r)/2 \rfloor$, $\lceil (b+r)/2 \rceil$. If $r > \lceil b/2 \rceil$, then the b-canonical cut is $r - \lfloor b/2 \rfloor$, b, $\lfloor b/2 \rfloor$. In both cases, the size of the cut is $\lfloor n/2 \rfloor \lceil n/2 \rceil$.

If $b \geq 2n/3$, then $q = \lfloor n/b \rfloor = 1$ and $r = n - q.b = n - b \leq n - 2n/3 = n/3 \leq b/2 \leq \lceil b/2 \rceil$, and the proof follows. $\square$

Lemma 2.6 (Similarity with complete graphs)**.** *Let $n \geq k+2$. The graph P_n^k has a maximum cut of size $\lfloor n/2 \rfloor \lceil n/2 \rceil$ if and only if $\lfloor 3n/2 \rfloor \leq 2k+2$. In such a case the $\lfloor n/2 \rfloor$-canonical cut of P_n^k is a maximum.*

Proof. First assume P_n^k has a cut $(S, \overline{S})$ of size $\lfloor n/2 \rfloor \lceil n/2 \rceil$, then every pair of nonadjacent vertices must be in the same part of the cut. Therefore, we can assume $\{v_1, v_{k+2}, v_{k+3}, \dots, v_n\} \in S$, and as a consequence $\{v_1, v_2, \dots, v_{n-k-1}, v_n\} \in S$. Since not all vertices can be in S, it implies $n-k-1 < k+2-1$, i.e. $n \leq 2k+1$. Observe that in such a case P_n^k has $2k+2-n$ universal vertices, and that these are the only vertices that have the option of being in $\overline{S}$. Hence, if we let x be the number of universal vertices in $\overline{S}$, we have that the size of the cut is $x(n-x)$, and since by hypothesis it is $\lfloor n/2 \rfloor \lceil n/2 \rceil$, then we obtain $x = \lfloor n/2 \rfloor$ or $x = \lceil n/2 \rceil$. Now, since $x \leq 2k+2-n$, the proof follows.

Conversely, if $\lfloor 3n/2 \rfloor \leq 2k+2$, then P_n^k has $n-2(n-(k+1)) = 2k+2-n \geq \lfloor 3n/2 \rfloor - n = \lfloor n/2 \rfloor$ universal vertices, and so it has a maximum cut of size $\lfloor n/2 \rfloor \lceil n/2 \rceil$. □

In Section 3, we will prove a stronger result for $k \leq 5$. We will prove that for every such k, there exists $b = b(k)$ such that the b-canonical cut of P_n^k is maximum for every n. In Section 4, we will prove that this result cannot be extended for $k = 6$. The proofs will consist in calculating the size of a canonical cut using previous Lemmas 2.2 and 2.3, and then comparing it with the size of an arbitrary maximum cut using the following Lemmas 2.7 and 2.8.

Given a cut C of P_n^k, we let c_i be the number of edges of C with an end vertex in $\{v_1, \dots, v_i\}$ and the other in $\{v_{i+1}, \dots, v_n\}$. Formally,

$$c_i = |C \cap (\{v_1, \dots, v_i\}, \{v_{i+1}, \dots, v_n\})|.$$

Lemma 2.7. *Let C be a maximum cut of P_n^k. If $1 \leq i \leq n-1$, then $mc(P_n^k) \leq mc(P_i^k) + c_i + mc(P_{n-i}^k)$.*

Proof. Let C be a maximum cut of P_n^k. The set C can be partitioned into three disjoint subsets: one containing the edges with both extremes in $\{v_1, \dots, v_i\}$; the other containing the edges with both extremes in $\{v_{i+1}, \dots, v_n\}$, and a third one containing the edges with an end vertex in $\{v_1, \dots, v_i\}$ and the other in $\{v_{i+1}, \dots, v_n\}$. Notice that the size of the latter set is c_i. Since the vertices $v_1, \dots, v_i$ induce a P_i^k and the vertices $v_{i+1}, \dots, v_n$ induce a P_{n-i}^k, we have that in the first and in the second set there are at most $mc(P_i^k)$ and $mc(P_{n-i}^k)$ edges, respectively. □

Lemma 2.8. *Let $C = (S, \overline{S})$ be a maximum cut of P_n^k. Then,*

1. *$c_1 \geq k/2$.*
2. *If $k \leq 4$, then $c_1 \leq k-1$. If $k = 3$, we may take $c_1 = 2$.*
3. *$c_1 \leq (2k+3)/3$, and we may take $c_1 < (2k+3)/3$.*
4. *If $v_1 \in S$ and $v_2 \in \overline{S}$, then*

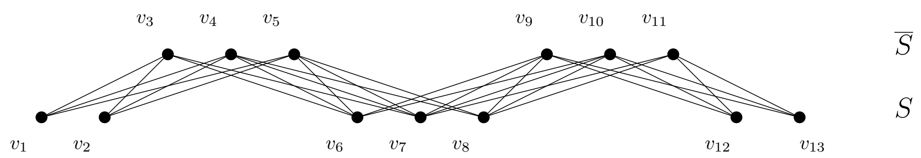

FIGURE 1. The 28 edges of the cut 2,3,3,3,2 of P_{13}^4

(a) $c_1 \leq \begin{cases} (k+3)/2 & \textit{if } v_{k+2} \in S; \\ (k+1)/2 & \textit{if } v_{k+2} \in \overline{S}. \end{cases}$

(b) $c_2 = \begin{cases} k & \textit{if } v_{k+2} \in S; \\ k-1 & \textit{if } v_{k+2} \in \overline{S}. \end{cases}$

(c) *In any case,* $c_1 \leq (k+3)/2$ *and* $c_2 \leq k$.

(d) *If* $c_1 > (k+1)/2$, *then* $c_2 = k$.

Proof. The first two statements follow trivially from the fact that every maximum cut is maximal, so if $v \in S$, then $|N(v) \cap S| \leq |N(v) \cap \overline{S}|$.

Assume $v_1 \in S$ and let j be the minimum i such that $v_i \in \overline{S}$. Clearly $j \leq k + 2 - c_1$. Since $|N(v_j) \cap \overline{S}| = |\{v_{j+1}, \ldots, v_{j+k}\} \cap \overline{S}| \geq c_1 - 1$, and $|N(v_j) \cap S| = j - 1 + |\{v_{j+1}, \ldots, v_{j+k}\} \cap S| = j - 1 + k - |\{v_{j+1}, \ldots, v_{j+k}\} \cap \overline{S}| \leq j - 1 + k - (c_1 - 1)$, we have $c_1 - 1 \leq j + k - c_1 \leq (k + 2 - c_1) + k - c_1$. Therefore, $c_1 \leq (2k+3)/3$. In addition if we consider C a maximum cut with the smallest c_1, we can assume $|N(v_j) \cap \overline{S}| < |N(v_j) \cap S|$ and so the proof follows.

Assume $v_1 \in S$ and $v_2 \in \overline{S}$. Let $t = |\{\overline{S} \cap \{v_3, \ldots, v_{k+1}\}|$. Looking at v_2 and its neighbors, in any maximum cut we have:

- If $v_{k+2} \in \overline{S}$, then $t + 1 \leq 1 + (k - 1 - t)$, therefore $t \leq (k-1)/2$. Since in this case $c_1 = t + 1$ and $c_2 = t + (k - 1 - t)$, the proof follows.
- If $v_{k+2} \in S$, then $t \leq 1 + (k - 1 - t) + 1$, therefore $t \leq (k+1)/2$. Since in this case $c_1 = t + 1$ and $c_2 = t + (k - 1 - t) + 1$, the proof follows.

Items c) and d) are direct consequences of the previous one. □

3. MAXIMUM CUTS FOR $k \leq 5$

We prove that there are b-canonical cuts that are maximum cuts for any $k \leq 5$. Actually, we establish that for $k \leq 5$, the value of b depends only on k and it is independent of n.

Theorem 3.1. *The 2-canonical cut of* P_n^2 *is maximum for all* $n \geq 4$, *and so*

$$mc(P_n^2) = \begin{cases} \lfloor n/2 \rfloor \lceil n/2 \rceil & \textit{if } 1 \leq n \leq 4; \\ \lfloor 3n/2 \rfloor - 2 & \textit{if } n \geq 4. \end{cases}$$

Proof. For $n \geq 4$, let $q = \lfloor n/2 \rfloor$ and $r = n - 2q$. If n is even, the 2-canonical cut C of P_n^2 is

$$1, \quad \underbrace{2,\, 2, \ldots,\, 2,}_{q-1} \quad 1.$$

If n is odd, it is

$$1, \quad \underbrace{2,\, 2, \ldots,\, 2,}_{q-1} \quad 2.$$

Thus, by Lemma 2.3, the size of C is $(q-2)a_{2,2} + 2a_{2,1}$ if n is even, and $(q-1)a_{2,2} + a_{2,1}$ if n is odd. By Lemma 2.2 with $k = 2$, $a_{2,1} = 2.1 = 2$; and $a_{2,2} = 2.(2-2) + \frac{1}{2}(2+2-2)(2-2+2+1) = 3$. It follows that $|C| = 3q - 2$ in the former case, and $|C| = 3q - 1$ in the latter. Notice that in both cases, it equals $\lfloor 3n/2 \rfloor - 2$. Thus, the size of the 2-canonical cut of P_n^2 is $\lfloor 3n/2 \rfloor - 2$ for every $n \geq 4$. To show that this is the maximum size of a cut, we proceed by induction on n. The base case $n = 4$ follows from Lemma 2.6; thus let $n > 4$ and assume that C' is maximum cut of P_n^2. By Lemma 2.7 and the inductive hypothesis,

$$|C'| \leq mc(P_1^2) + c_1' + mc(P_{n-1}^2) = \lfloor 3n/2 \rfloor - 2 + (c_1' + \lfloor 3(n-1)/2 \rfloor - \lfloor 3n/2 \rfloor).$$

Since $\lfloor 3(n-1)/2 \rfloor - \lfloor 3n/2 \rfloor \in \{-1, -2\}$, if $c_1' \leq 1$ we are done. Then let $c_1' = 2$, which implies $c_2' = 2$ by Lemma 2.8. Therefore, by Lemma 2.7 and the inductive hypothesis if $n > 5$, we have

$$|C'| \leq mc(P_2^2) + c_2' + mc(P_{n-2}^2) = 1 + 2 + \lfloor 3(n-2)/2 \rfloor - 2 = \lfloor 3n/2 \rfloor - 2.$$

If $n = 5$, we have

$$|C'| \leq mc(P_2^2) + c_2' + mc(P_3^2) = 1 + 2 + 1.2 = 5 = \lfloor 3n/2 \rfloor - 2.$$

□

Theorem 3.2. *The 2-canonical cut of P_n^3 is maximum for all $n \geq 5$, and so*

$$mc(P_n^3) = \begin{cases} \lfloor n/2 \rfloor \lceil n/2 \rceil & \text{if } 1 \leq n \leq 5; \\ 2n - 4 & \text{if } n \geq 5. \end{cases}$$

Proof. The first part of this proof is the same as that of the previous theorem with the only difference that here $k = 3$ and so $a_{2,1} = 2.1 = 2$ and $a_{2,2} = 2.2 = 4$. Then, the size of the 2-canonical cut C of P_n^3 is $|C| = 4(q-2) + 2.2 = 4q - 4$ if n is even, and $|C| = 4(q-1) + 2 = 4q - 2$ if n is odd. Observe that in both cases $|C| = 2n - 4$.

To prove that this is the maximum size of a cut of P_n^3, we will proceed by induction on n as in the previous theorem. The base case $n = 5$ follows from Lemma 2.6. Let $n > 5$ and assume C' is a maximum cut of P_n^3. By Lemma 2.8, we also can assume $c_1' = 2$. By Lemma 2.7 and the inductive hypothesis,

$$|C'| \leq mc(P_1^3) + c_1' + mc(P_{n-1}^3) = 2 + 2(n-1) - 4 = 2n - 4.$$

□

Theorem 3.3. *The 3-canonical cut of P_n^4 is maximum for all $n \geq 6$, and so*

$$mc(P_n^4) = \begin{cases} \lfloor n/2 \rfloor \lceil n/2 \rceil & \text{if } 1 \leq n \leq 6; \\ 3n - 7 - \lfloor (n+1)/3 \rfloor & \text{if } n \geq 6. \end{cases}$$

Proof. For $n \geq 6$, let $q = \lfloor n/3 \rfloor$ and $r = n - 3q$. If $r = 0$, the 3-canonical cut C of P_n^4 is

$$1, \underbrace{3, 3, ..., 3,}_{q-1} \; 2.$$

If $r = 1$, it is

$$2, \underbrace{3, 3, ..., 3,}_{q-1} \; 2.$$

And if $r = 2$, it is

$$2, \underbrace{3, 3, ..., 3,}_{q-1} \; 3.$$

Thus, by Lemma 2.3, in each case the size of C is $(q-2)a_{3,3} + a_{3,1} + a_{3,2}$; $(q-2)a_{3,3} + 2a_{3,2}$, and $(q-1)a_{3,3} + a_{3,2}$, respectively. By Lemma 2.2 with $k = 4$, $a_{3,1} = 3.1 = 3$; $a_{3,2} = 3.2 = 6$ and $a_{3,3} = 3(4-3) + \frac{1}{2}(3+3-4)(3-3+4+1) = 8$. It follows that $|C| = (q-2)8 + 3 + 6 = 8q - 7$ if $r = 0$; $|C| = (q-2)8 + 2.6 = 8q - 4$ if $r = 1$; and $|C| = (q-1)8 + 6 = 8q - 2$ if $r = 2$. Notice that in any case $|C| = 3n - 7 - \lfloor (n+1)/3 \rfloor$. Thus, the size of the 3-canonical cut of P_n^4 is $3n - 7 - \lfloor (n+1)/3 \rfloor$ for every $n \geq 6$.

To show that this is the maximum size of a cut of P_n^4, we will proceed again by induction on n. The base case $n = 6$ follows from Lemma 2.6. Let $n > 6$ and assume C' is a maximum cut of P_n^4. By Lemma 2.7 and the inductive hypothesis,

$$|C'| \leq mc(P_1^4) + c'_1 + mc(P_{n-1}^4) = c'_1 + 3(n-1) - 7 - \lfloor n/3 \rfloor = \\ 3n - 7 - \lfloor (n+1)/3 \rfloor + (c'_1 - 3 + \lfloor (n+1)/3 \rfloor - \lfloor n/3 \rfloor).$$

Since $\lfloor (n+1)/3 \rfloor - \lfloor n/3 \rfloor \in \{0, 1\}$ and, by Lemma 2.8, $c'_1 \leq k - 1 = 3$; it follows that if $\lfloor (n+1)/3 \rfloor - \lfloor n/3 \rfloor = 0$, we are done. Hence we can assume $\lfloor (n+1)/3 \rfloor - \lfloor n/3 \rfloor = 1$ and $c'_1 = 3$. Notice that the former assumption implies $\lfloor (n+1)/3 \rfloor - \lfloor (n-1)/3 \rfloor = 1$, and the latter implies $c'_2 = k = 4$ by Lemma 2.8. Therefore, applying Lemma 2.7 and the inductive hypothesis if $n \geq 8$, we have

$$|C'| \leq mc(P_2^4) + c'_2 + mc(P_{n-2}^4) = 1 + 4 + 3(n-2) - 7 - \lfloor (n-1)/3 \rfloor = \\ 5 + 3n - 6 - 7 - (\lfloor (n+1)/3 \rfloor - 1) = 3n - 7 - \lfloor (n+1)/3 \rfloor.$$

If $n = 7$, we have

$$|C'| \leq mc(P_2^4) + c_2 + mc(P_{7-2}^4) \leq 1 + 4 \\ + 2.3 = 11 \leq 12 = 3.n - 7 - \lfloor (n+1)/3 \rfloor.$$

$\square$

Theorem 3.4. *The 4-canonical cut of P_n^5 is maximum for all $n \geq 7$, and so*

$$mc(P_n^5) = \begin{cases} \lfloor n/2 \rfloor \lceil n/2 \rceil & \textit{if } 1 \leq n \leq 7; \\ 3n + \lfloor n/4 \rfloor - 10 & \textit{if } n \geq 7. \end{cases}$$

Proof. For $n \geq 7$, let $q = \lfloor n/4 \rfloor$ and $r = n - 4q$. If $r = 0$, the 4-canonical cut C of P_n^5 is

$$2, \underbrace{4, 4, ..., 4,}_{q-1} 2.$$

If $r = 1$, it is

$$2, \underbrace{4, 4, ..., 4,}_{q-1} 3.$$

If $r = 2$, it is

$$3, \underbrace{4, 4, ..., 4,}_{q-1} 3.$$

And if $r = 3$, it is

$$1, \underbrace{4, 4, 4, ..., 4,}_{q} 2.$$

Thus, by Lemma 2.3, in each case the size of C is $(q-2)a_{4,4} + 2a_{4,2}$; $(q-2)a_{4,4} + a_{4,2} + a_{4,3}$; $(q-2)a_{4,4} + 2a_{4,3}$, and $(q-1)a_{4,4} + a_{4,1} + a_{4,2}$, respectively. By Lemma 2.2 with $k = 5$, $a_{4,1} = 4.1 = 4$; $a_{4,2} = 4.2 = 8$; $a_{4,3} = 4.(5-4) + \frac{1}{2}.(4+3-5).(4-3+5+1) = 11$; and $a_{4,4} = 4.(5-4) + \frac{1}{2}.(4+4-5).(4-4+5+1) = 13$. It follows that $|C| = (q-2)13 + 2.8 = 13q - 10$ if $r = 0$; $|C| = (q-2)13 + 8 + 11 = 13q - 7$ if $r = 1$; $|C| = (q-2)13 + 2.11 = 13q - 4$ if $r = 2$; and $|C| = (q-1)13 + 4 + 8 = 13q - 1$ if $r = 3$. Notice that in any case $|C| = 3n + \lfloor n/4 \rfloor - 10$. Thus, the size of the 4-canonical cut of P_n^5 is $3n + \lfloor n/4 \rfloor - 10$ for every $n \geq 7$.

To prove that this is the maximum size of a cut of P_n^5, we will proceed again by induction on n. The base case $n = 7$ follows from Lemma 2.6. Let $n > 7$ and assume C' is a maximum cut of P_n^5. By Lemma 2.7 and the inductive hypothesis,

$$|C'| \leq mc(P_1^5) + c_1' + mc(P_{n-1}^5) = c_1' + 3(n-1) + \lfloor (n-1)/4 \rfloor - 10 = \\ 3n + \lfloor n/4 \rfloor - 10 + (c_1' + \lfloor (n-1)/4 \rfloor - \lfloor n/4 \rfloor - 3)).$$

Since $\lfloor (n-1)/4 \rfloor - \lfloor n/4 \rfloor \in \{0, -1\}$, and, by Lemma 2.8, $c_1' \leq k - 1 = 4$; it follows that if $c_1' < 4$, or if $c_1' = 4$ and $\lfloor (n-1)/4 \rfloor - \lfloor n/4 \rfloor = -1$, then we are done. Thus we can assume $c_1' = 4$ and $\lfloor (n-1)/4 \rfloor - \lfloor n/4 \rfloor = 0$. By Lemma 2.8, $c_2' = 4$. Therefore, applying Lemma 2.7 and the inductive hypothesis if $n \geq 9$, we have

$$|C'| \leq mc(P_2^5) + c_2' + mc(P_{n-2}^5) = 1 + 4 + 3(n-2) + \lfloor (n-2)/4 \rfloor - 10 = \\ 3n + \lfloor n/4 \rfloor - 10 + (\lfloor (n-2)/4 \rfloor - \lfloor n/4 \rfloor - 1) \leq 3n + \lfloor n/4 \rfloor - 10.$$

If $n = 8$, we have

$$|C'| \leq mc(P_2^5) + c_2' + mc(P_6^5) = 1+4+3(8-2)+\lfloor (8-2)/4 \rfloor - 10 = 14 \leq 16 = 3n + \lfloor n/4 \rfloor - 10.$$

□

4. The case $k = 6$

We let $C_{n,k;b}$ denote the b-canonical cut of P_n^k. In Lemma 4.1 below, we have calculated the size of $C(n, 6, b)$ for all possible values of n and $2 \leq b \leq 6$. The proof is omitted because it is completely analogous to the theorems of the previous section. We introduce a new notation to simplify the writing: for a given n and b, we will use a vector $(x_0, x_1, \ldots, x_{b-1})_n^b$ to indicate the value that must be considered depending on the remainder of n divided by b.

Lemma 4.1. *For every $n \geq 8$, we have that*

$$\begin{aligned}
|C_{n,6;2}| &= 3\lfloor n/2 \rfloor + (-2, -1)_n^2; \\
|C_{n,6;3}| &= 9\lfloor n/3 \rfloor + (-9, -6, -3)_n^3; \\
|C_{n,6;4}| &= 15\lfloor n/4 \rfloor + (-14, -10, -6, -3)_n^4; \\
|C_{n,6;5}| &= 19\lfloor n/5 \rfloor + (-14, -10, -7, -4, 1)_n^5 \text{ ; and} \\
|C_{n,6;6}| &= 21\lfloor n/6 \rfloor + (-12, -9, -6, -4, 0, 5)_n^6.
\end{aligned}$$

Using the previous lemma, we have completed the following table.

n	$\lvert c_{n,6;2}\rvert$	$\lvert c_{n,6;3}\rvert$	$\lvert c_{n,6;4}\rvert$	$\lvert C_{n,6;5}\rvert$	$\lvert C_{n,6;6}\rvert$
8	10	15	**16**	15	15
9	11	18	**20**	**20**	17
10	13	21	**24**	**24**	21
11	14	24	27	**28**	26
12	16	27	**31**	**31**	30
13	17	30	**35**	34	33
14	19	33	**39**	**39**	36
15	20	36	42	**43**	38
16	22	39	46	**47**	42
17	23	42	**50**	**50**	47
18	25	51	**54**	53	51
19	26	48	57	**58**	54
20	28	51	61	**62**	57

Observe that, in contrast to the cases $k \leq 5$, the value of b that maximizes $|C_{n,6;b}|$ depends on n, for instance: the best canonical cut of P_8^6 is obtained by taking $b = 4$; whereas for P_{11}^6, it is obtained by taking $b = 5$. Thus we have the following result.

Lemma 4.2. *For $k = 6$, it does not exist a unique value for b such that the b-canonical cut of P_n^6 is the best canonical cut for all $n \geq 8$.*

We verified an asymptotic behavior, for $k = 6$. Using Lemma 4.1, in what follows, we prove that for n big enough, the size of the 5-canonical cut of P_n^6 is larger than the size of any other canonical cut.

Lemma 4.3. *For $n \geq 524$, the best canonical cut of P_n^6 is always realized by $b = 5$.*

Proof. For $n \geq 8$, we have
$|C_{n,6;6}| - |C_{n,6;5}| = 21\lfloor n/6 \rfloor + (-12,-9,-6,-4,0,5)_n^6 - (19\lfloor n/5 \rfloor + (-14,-10,-7,-4,1)_n^5) = 21\lfloor n/6 \rfloor - 19\lfloor n/5 \rfloor + (-12,-9,-6,-4,0,5)_n^6 + (14,10,7,4,-1)_n^5 \leq 21\lfloor n/6 \rfloor - 19\lfloor n/5 \rfloor + (5+14) = 21\lfloor n/6 \rfloor - 19\lfloor n/5 \rfloor + 19 \leq 21(n/6) - 19((n-5+1)/5) + 19 = -3n/10 + 171/5.$

Therefore $|C_{n,6;5}| \geq |C_{n,6;6}|$ whenever $n \geq 114$. In addition,
$|C_{n,6;5}| - |C_{n,6;4}| = 19\lfloor n/5 \rfloor + (-14,-10,-7,-4,1)_n^5 - (15\lfloor n/4 \rfloor + (-14,-10,-6,-3)_n^4) = 19\lfloor n/5 \rfloor - 15\lfloor n/4 \rfloor + (-14,-10,-7,-4,1)_n^5 + (14,10,6,3)_n^4 \geq 19\lfloor n/5 \rfloor - 15\lfloor n/4 \rfloor + (-14+3) = 19\lfloor n/5 \rfloor - 15\lfloor n/4 \rfloor - 11 \geq 19((n-5+1)/5) - 15(n/4) - 11 = n/20 - 131/5.$

Thus $|C_{n,6;5}| \geq |C_{n,6;4}|$ whenever $n \geq 524$. Also,
$|C_{n,6;5}| - |C_{n,6;3}| = 19\lfloor n/5 \rfloor + (-14,-10,-7,-4,1)_n^5 - (9\lfloor n/3 \rfloor + (-9,-6,-3)_n^3) = 19\lfloor n/5 \rfloor - 9\lfloor n/3 \rfloor + (-14,-10,-7,-4,1)_n^5 + (9,6,3)_n^3 \geq 19\lfloor n/5 \rfloor - 9\lfloor n/3 \rfloor + (-14+3) = 19\lfloor n/5 \rfloor - 9\lfloor n/3 \rfloor - 11 \geq 19((n-5+1)/5) - 9(n/3) - 11 = 4n/5 - 131/5.$

It implies $|C_{n,6;5}| \geq |C_{n,6;3}|$ whenever $n \geq 131/4 \geq 32$. And finally,
$|C_{n,6;5}| - |C_{n,6;2}| = 19\lfloor n/5 \rfloor + (-14,-10,-7,-4,1)_n^5 - (3\lfloor n/2 \rfloor + (-2,-1)_n^2) = 19\lfloor n/5 \rfloor - 3\lfloor n/2 \rfloor + (-14,-10,-7,-4,1)_n^5 + (2,1)_n^2 \geq 19\lfloor n/5 \rfloor - 3\lfloor n/2 \rfloor + (-14+1) \geq 19((n-5+1)/5) - 3(n/2) - 13 = 23n/10 - 141/5.$

Therefore $|C_{n,6;5}| \geq |C_{n,6;2}|$ whenever $n \geq 282/23 \geq 12$. □

5. Reduced co-bipartite chains

Recall that a path power graph P_n^k is a co-bipartite chain if and only if $n \leq 2k+2$. When $n = 2k+2$, the graph is reduced and does not have universal vertices. When $n = 2k+1$, the graph is reduced and has exactly one universal vertex.

In the abstract of [7], Boyaci et al. say that they have determined an explicit formula for the size of the maximum cut of a twin-free (reduced) co-bipartite chain graph. However, in the statement of Theorem 2 where that issue is considered, the expression that gives the size of the maximum cut is not an explicit exact formula. Nevertheless, in the first part of the proof of that theorem, the authors obtain the formula $mc(P_{2k+2}^k) = \lfloor \frac{5}{6}k^2 - \frac{3}{2}k + \frac{3}{4} \rfloor$. This expression clearly contains an error since it is not in accordance with the size of the cut given by the same authors for P_{12}^5 in Figure 1 of that paper. Following that proof, the error (probably a typo) can be discovered, it should say $mc(P_{2k+2}^k) = \lfloor \frac{5}{6}k^2 + \frac{3}{2}k + \frac{3}{4} \rfloor$ (a + instead of a -). In the second part of the proof of that theorem, the authors obtain all the necessary elements to calculate the size of a maximum cut of P^k_{2k+1}, but it is not formulated explicitly. They proved that $mc(P^k_{2k+1}) = 2x(y+z)+z(x+y)+\frac{y(y+1)}{2}+yz+2x$, where $(x, y, z) = (\frac{k}{3}+\frac{1}{2}, \frac{k}{3}-\frac{1}{3}, \frac{k}{3}-\frac{1}{6}) + (\delta_x, \delta_y, \delta_z)$, and

$$(\delta_x, \delta_y, \delta_z) = \begin{cases} (-1/2, 1/3, 1/6), & \text{if } k \equiv 0 \pmod 3; \\ (1/6, 0, -1/6), & \text{if } k \equiv 1 \pmod 3; \\ (-1/6, -1/3, 1/2), & \text{if } k \equiv 2 \pmod 3) . \end{cases}$$

Finishing those calculations we have obtained that

$$(x, y, z) = \begin{cases} (k/3, k/3, k/3), & \text{if } k \equiv 0 \pmod 3; \\ (k/3+2/3, k/3-1/3, k/3-1/3), & \text{if } k \equiv 1 \pmod 3; \\ (k/3+1/3, k/3-2/3, k/3+1/3), & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Therefore,

$$mc(P^k_{2k+1}) = \begin{cases} (5k^2+5k)/6, & \text{if } k \equiv 0 \pmod 3; \\ (5k^2+5k+2)/6, & \text{if } k \equiv 1 \pmod 3; \\ (5k^2+5k)/6, & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Next, we will use those results to answer our question for the path powers that are reduced co-chain, i.e. for P^k_{2k+2} and P^k_{2k+1}.

Theorem 5.1. *For a positive integer $k \geq 6$, let $b = 2\lfloor\frac{k}{3}\rfloor + (1, 1, 2)^3_k$. The b-canonical cut $C_{2k+2,k;b}$ of P^k_{2k+2} is maximum, and so*

$$mc(P^k_{2k+2}) = |C_{2k+2,k;b}| = \frac{5}{6}k^2 + \frac{9}{6}k + (0, \frac{2}{3}, \frac{2}{3})^3_k.$$

Proof. First we calculate the size of the b-canonical cuts, let $r_b(n) = n - \lfloor n/b\rfloor b$:

Case 1: $k = 3m$.

Then $b = 2m + 1$, $q = \lfloor(2k + 2)/b\rfloor = 2$ and $r_b(2k + 2) = 2m > m + 1 = \lceil b/2\rceil$, then the canonical cut $C_{2k+2,k;b}$ is m, $2m+1$, $2m+1$, m. By Lemma 2.2, in this case we have $a_{2m+1,m} = (2m + 1)m$ and $a_{2m+1,2m+1} = (2m+1)^2 - ((m+2)(m+1)/2)$. Therefore, by Lemma 2.3, $|C_{2k+2,k;b}| = (15m^2 + 9m)/2$.

Case 2: $k = 3m + 1$.

Then $b = 2m+1$, $q = \lfloor(2k+2)/b\rfloor = 3$ and $r_b(2k+2) = 1 \leq m+1 = \lceil b/2\rceil$, then the canonical cut $C_{2k+2,k;b}$ is $m + 1$, $2m + 1$, $2m + 1$, $m + 1$. By Lemma 2.2, in this case we have $a_{2m+1,m+1} = (2m + 1)(m + 1)$, and $a_{2m+1,2m+1} = (2m+1)m + ((m+1)(3m+2)/2)$. Therefore, by Lemma 2.3, $|C_{2k+2,k;b}| = (15m^2 + 19m + 6)/2$.

Case 3: $k = 3m + 2$.

Then $b = 2m + 2$, $q = \lfloor(2k + 2)/b\rfloor = 3$ and $r_b(2k + 2) = 0 \leq \lceil b/2\rceil$, then the canonical cut $C_{2k+2,k;b}$ is $m + 1$, $2m + 2$, $2m + 2$, $m + 1$. By Lemma 2.2, in this case we have $a_{2m+2,m+1} = (2m + 2)(m + 1)$, and

$a_{2m+2,2m+2} = (2m+2)^2 - ((m+2)(m+1)/2)$. Therefore, by Lemma 2.3, $|C_{2k+2,k;b}| = (15m^2 + 29m + 14)/2$.

Now we verify that the sizes of the canonical cuts equal the sizes obtained in [7]:

If $k = 3m$, then $\lfloor \frac{5}{6}k^2 + \frac{3}{2}k + \frac{3}{4} \rfloor = \lfloor \frac{5}{6}(3m)^2 + \frac{3}{2}(3m) + \frac{3}{4} \rfloor = (15m^2 + 9m)/2$.

If $k = 3m+1$, then $\lfloor \frac{5}{6}k^2 + \frac{3}{2}k + \frac{3}{4} \rfloor = \lfloor \frac{5}{6}(3m+1)^2 + \frac{3}{2}(3m+1) + \frac{3}{4} \rfloor = (15m^2 + 19m + 6)/2$.

If $k = 3m+2$, then $\lfloor \frac{5}{6}k^2 + \frac{3}{2}k + \frac{3}{4} \rfloor = \lfloor \frac{5}{6}(3m+2)^2 + \frac{3}{2}(3m+2) + \frac{3}{4} \rfloor = (15m^2 + 29m + 14)/2$. □

Theorem 5.2. *For a positive integer $k \geq 6$, let $b = 2\lfloor \frac{k}{3} \rfloor + (1,1,2)^3_k$. The b-canonical cut $C_{2k+1,k;b}$ of P^k_{2k+1} is maximum, and so*

$$mc(P^k_{2k+1}) = |C_{2k+1,k;b}| = \frac{5}{6}k^2 + \frac{5}{6}k + (0, \frac{1}{3}, 0)^3_k.$$

Proof. First we calculate the size of the b-canonical cuts, let $r_b(n) = n - \lfloor n/b \rfloor b$:

Case 1: $k = 3m$.

Then $b = 2m$, $q = \lfloor (2k+1)/b \rfloor = 3$ and $r_b(2k+1) = 1 \leq m = \lceil b/2 \rceil$, then the canonical cut $C_{2k+1,k;b}$ is m, $2m$, $2m$, $m+1$. By Lemma 2.2, in this case we have $a_{2m,m} = 2m^2$, $a_{2m,m+1} = 2m(m+1)$ and $a_{2m,2m} = (7m^2 + m)/2$. Therefore, by Lemma 2.3, $|C_{2k+1,k;b}| = (15m^2 + 5m)/2$.

Case 2: $k = 3m+1$.

Then $b = 2m+1$, $q = \lfloor (2k+1)/b \rfloor = 3$ and $r_b(2k+1) = 0 \leq m+1 = \lceil b/2 \rceil$, then the canonical cut $C_{2k+1,k;b}$ is m, $2m+1$, $2m+1$, $m+1$. By Lemma 2.2, in this case we have $a_{2m+1,m} = (2m+1)m$, $a_{2m+1,m+1} = (2m+1)(m+1)$, and $a_{2m+1,2m+1} = (7m^2 + 7m + 2)/2$. Therefore, by Lemma 2.3, $|C_{2k+1,k;b}| = (15m^2 + 15m + 4)/2$.

Case 3: $k = 3m+2$.

Then $b = 2m+2$, $q = \lfloor (2k+1)/b \rfloor = 2$ and $r_b(2k+1) = 2m+1 > m+1 = \lceil b/2 \rceil$, then the canonical cut $C_{2k+1,k;b}$ is m, $2m+2$, $2m+2$, $m+1$. By Lemma 2.2, in this case we have $a_{2m+2,m} = (2m+2)m$, $a_{2m+2,m+1} = (2m+2)(m+1)$, and $a_{2m+2,2m+2} = (7m^2 + 13m + 6)/2$. Therefore, by Lemma 2.3, $|C_{2k+2,k;b}| = (15m^2 + 25m + 10)/2$.

Now we verify that the sizes of the canonical cuts equal the sizes obtained in [7]:

If $k = 3m$, then $(5k^2 + 5k)/6 = \frac{5}{6}(k^2 + k) = \frac{5}{6}(9m^2 + 3m) = (15m^2 + 5m)/2$.

If $k = 3m+1$, then $(5k^2 + 5k + 2)/6 = (5(3m+1)^2 + 5(3m+1) + 2)/6 = (15m^2 + 15m + 4)/2$.

If $k = 3m+2$, then $(5k^2 + 5k)/6 = \frac{5}{6}(k^2 + k) = \frac{5}{6}((3m+2)^2 + 3m + 2) = (15m^2 + 25m + 10)/2$. □ □

6. Computational evidence and concluding remarks

To examine whether the canonical cuts give us the best cuts, we did several computational experiments. We have computed the maximum cut of P^k_n on the range $1 \leq k \leq n \leq 43$ using the Gurobi software [12] assigning Boolean variables x_i for $1 \leq i \leq n$ and the objective function

$$\sum_{(u,v)\in E(G)} (x_u - x_v)^2.$$

The graphs P_n^k that we considered are very dense which impose a limit on Gurobi, whose computation stops because of memory limitations when a certain value of k is reached. We also used SageMath [17], but its range of applicability was smaller than Gurobi. We also used the MaxSat solver EvalMaxSat, but again the range of applicability was smaller than Gurobi. Whenever more than one result was available, we checked for consistency and did not find any incoherency. We could not find a free software that is as effective as Gurobi for solving the MaxCut. All codes used are available at MaxCut Codes Repository [18].

n	k	max cut size (sequence)	best canon. cut size (b)	n	k	max cut size (sequence)	best canon. cut size (b)
16	8	55 (3,6,5,2)	54 (5)	37	19	292 (7,14,12,4)	291 (13)
22	8	85 (1,5,6,6,4)	84 (6)	37	20	301 (2,11,15,9)	300 (15)
22	11	103 (2,7,8,5)	102 (7)	38	6	130 (2,4,5,5,5,5,5,5,2)	129 (4)
23	6	73 (1,4,5,5,5,3)	72 (4)	38	11	211 (4,8,8,8,7,3)	210 (8)
23	12	115 (5,9,7,2)	114 (8)	38	14	251 (5,10,10,9,4)	250 (10)
25	13	135 (1,7,10,7)	134 (9)	38	20	312 (9,15,12,2)	310 (13)
27	14	157 (6,11,8,2)	156 (9)	39	9	186 (4,6,6,6,7,6,4)	185 (6)
28	6	92 (3,5,5,4,4,4,3)	91 (4)	39	10	203 (5,7,8,8,7,4)	202 (7)
28	8	115 (3,6,6,6,5,2)	114 (6)	39	20	324 (5,12,14,8)	323 (13)
28	14	166 (6,10,9,3)	165 (9)	39	21	333 (1,11,16,11)	332 (16)
29	15	181 (7,11,9,2)	180 (10)	40	8	175 (2,5,6,6,6,6,6,3)	174 (6)
30	11	157 (5,8,8,7,2)	156 (8)	40	9	192 (4,6,6,6,7,7,4)	191 (7)
30	16	197 (7,12,9,2)	196 (12)	40	15	281 (7,11,11,9,2)	280 (10)
31	16	206 (7,12,10,2)	205 (11)	40	20	337 (8,14,13,5)	336 (13)
32	12	181 (1,7,9,9,6)	180 (8)	40	21	345 (3,13,15,9)	343 (14)
32	17	223 (9,13,10)	222 (13)	41	12	247 (5,9,9,9,7,2)	246 (8)
33	6	111 (2,5,4,5,5,4,5,3)	110 (4)	41	21	358 (9,15,13,4)	357 (14)
33	9	153 (4,7,7,7,6,2)	152 (6)	41	22	367 (2,13,16,10)	366 (17)
33	17	233 (7,13,10,3)	232 (11)	42	16	313 (8,12,12,10)	312 (12)
34	8	145 (3,6,6,6,6,5,2)	144 (6)	42	22	380 (4,13,16,9)	377 (14)
34	17	244 (5,11,12,6)	243 (11)	43	6	149 (2,4,5,5,4,5,5,5,5,3)	148 (5)
34	18	251 (1,10,14,9)	250 (14)	43	12	262 (4,9,9,8,9,4)	261 (8)
35	18	262 (3,11,13,8)	261 (12)	43	22	393 (9,16,14,4)	392 (15)
36	19	281 (2,11,14,9)	279 (15)	43	23	403 (3,13,17,10)	401 (18)

TABLE 1. Pairs $(n, k), 1 \leq k \leq n \leq 43$, for which there is no b-canonical cut with same size of the maximum cut of P_n^k. For each pair (n, k), the size of a maximum cut and a best b-canonical cut size is exhibited.

In Table 1 we describe the experiment data. In Table 1 we display the values of k and n for which there is no canonical cut of P_n^k that is maximum. We depict the corresponding value of b, such that the b-canonical is the best approximation for the maximum cut. We checked that for the values of n and k not appearing at Table 1 there is a b-canonical cut with size equal to the size of the maximum cut of P_n^k.

Characterizing the cases in which the class P_n^k admits maximum cuts that are canonical, in addition to being a combinatorial challenge, also provides as a by-product a formula to compute the maximum cut as a function of n and k, which allows determining the maximum cut size from the integers n and k without having to construct and traverse the graph P_n^k.

We observe that, in our experiment with $n \leq 43$, the difference between the canonical and the maximum cut sizes of P_n^k is at most 3. Hence, our experiments have shown that canonical cuts are a nice strategy to find good approximations for the maximum cut on power of path graphs P_n^k. Surprisingly, the only missing value of k is 7, so the computational experiments suggest that for $k = 7$ the maximum cut is canonical. In a b-canonical cut we have n/b blocks of size b and we know that $a_{b,b} = b(k-b)+(2b-k)(k+1)/2$, so looking for the best b we can try to maximize $a_{b,b}(n/b) = (k-b)+(2-k/b)(k+1)/2$. This function has derivative $-1 + (k+1)k/2b^2$, so it has a maximum at $b = \sqrt{k(k+1)/2}$. At the moment the best canonical cuts we have obtained are with b around this number.

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