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The P versus NP–complete dichotomy of some challenging problems in graph theory☆

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ABSTRACT

The Clay Mathematics Institute has selected seven Millennium Problems to motivate research on important classic questions that have resisted solution over the years. Among them is the central problem in theoretical computer science: the P versus NP problem, which aims to classify the possible existence of efficient solutions to combinatorial and optimization problems. The main goal is to determine whether there are questions whose answer can be quickly checked, but which require an impossibly long time to solve by any direct procedure. In this context, it is important to determine precisely what facet of a problem makes it NP-complete. We shall discuss classes of problems for which dichotomy results do exist: every problem in the class is classified into polynomial or NP-complete. We shall discuss our contribution through the classification of some long-standing problems in important areas of graph theory: perfect graphs, intersection graphs, and structural characterization of graph classes. More precisely, we have shown that Chvátal's SKEW PARTITION is polynomial and that Roberts–Spencer's CLIQUE GRAPH is NP-complete. We have also solved the dichotomy for Golumbic–Kaplan–Shamir's SANDWICH problem. We shall describe two examples where we can determine the full dichotomy: the edge-colouring problem for graphs with no cycle with a unique chord and the three nonempty part sandwich problem. Some open problems are discussed: the stubborn problem for list partition, the chromatic index of chordal graphs, and the recognition of split clique graphs.

1. Overview

One of the seven Millennium Problems[1] selected by the Clay Mathematics Institute is the P versus NP problem, a central problem in theoretical computer science [29]:

> *Are there questions whose answer can be quickly checked, but which require an impossibly long time to solve by any direct procedure?*

My personal approach to the P versus NP problem is to classify challenging combinatorial problems into P or NP-complete. This paper presents our contribution to graph theory: the classification of two long-standing problems, one into P, and the other into NP-complete; the definition of two full dichotomies, of two classes of problems for which every problem is classified into P or NP-complete.

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[1] It is remarkable that the first Clay Mathematics Institute Millennium prize has just been announced on 18 March 2010, awarded to Grigory Perelman for resolution of the Poincaré conjecture; see http://www.claymath.org/ for details.

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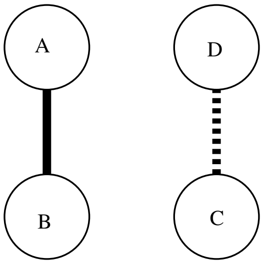

Fig. 1. A, B, C, D is a skew partition, and $A \cup B$ is a skew cutset.

Section 2 discusses the classification of the SKEW PARTITION problem as polynomial and of the CLIQUE GRAPH problem as NP-complete. Section 3 discusses the search for interesting problems and interesting graph classes, and the concept of complexity-separating problems and classes. Section 4 discusses the definition of two full dichotomies: one for sandwich problems and one for edge-colouring. Throughout the paper, several related problems are proposed, and in Section 5 we conclude by collecting the main proposed complexity-separating questions.

2. Two long-standing problems in graph theory

Two long-standing problems in graph theory are the SKEW PARTITION problem and the CLIQUE GRAPH problem. The SKEW PARTITION problem was defined by Chvátal in 1985 as the recognition problem for a special decomposition arising in the context of perfect graphs [18]. The CLIQUE GRAPH problem was defined by Roberts and Spencer in 1971 as the recognition problem for a class of graphs arising in the context of intersection graphs [56].

Both problems have been much studied, and are discussed in several graph theory books [5,10,43,51,53,64]. The SKEW PARTITION plays a central role in the solution of the Perfect Graph Theorem [15,17]. Reed [55] gives a good account of the introduction of SKEW PARTITION in connection with perfect graphs. The CLIQUE GRAPH is widely studied in Graph Dynamics, graph operators, the clique operator and its image [48,52,62].

It is additionally remarkable that both SKEW PARTITION and CLIQUE GRAPH were proved to be in NP when their classification into P or NP-complete was proposed [18,56]. The long-standing challenge was the search for a polynomial-time algorithm or an NP-completeness proof.

2.1. Skew partition

A *skew partition* in a graph $G = (V, E)$ is a partition of the vertex set V into four nonempty parts A, B, C, D such that there are all possible edges between parts A and B but no edges between parts C and D. (See Fig. 1.) By definition, in G, the set $A \cup B$ is a cutset whose removal disconnects nonempty parts C and D, and, in the complement of G, the set $C \cup D$ is a cutset whose removal disconnects nonempty parts A and B. This is why Chvátal [18] called $A \cup B$ a *skew cutset* and A, B, C, D a skew partition. The concept of a skew partition generalizes well-studied decompositions: star cutset, clique cutset, and homogeneous set.

The first polynomial-time algorithm for testing whether a graph admits a skew partition was obtained in collaboration with Sulamita Klein, Yoshiharu Kohayakawa, and Bruce Reed [26]. The polynomial-time algorithm actually solves the more general *list skew partition problem*, where the input contains, for each vertex, a list containing some of the four parts. The two decision problems are stated as follows.

SKEW PARTITION
Instance: Graph $G = (V, E)$.
Question: Does V admit a partition into four nonempty parts A, B, C, D such that each vertex in A is adjacent to each vertex in B and each vertex in C is nonadjacent to each vertex in D?

LIST SKEW PARTITION
Instance: Graph $G = (V, E)$ and, for each $v \in V$, a list $L(v) \subseteq \{A, B, C, D\}$.
Question: Does V admit a partition into four parts A, B, C, D such that each vertex in A is adjacent to each vertex in B, each vertex in C is nonadjacent to each vertex in D, and such that each vertex v is assigned to a part in $L(v)$?

Note that a skew partition requires its four parts to be nonempty whereas a list skew partition does not require its four parts to be nonempty. Refer to Fig. 2, where instance G, L admits a list skew partition by setting the two bottom vertices into part A and the two top vertices into part C, and parts B and D to be empty. On the other hand, the graph G itself admits no skew partition.

The LIST SKEW PARTITION was proposed by Feder, Hell, Klein, and Motwani in a seminal paper [23] in which list partitions were introduced and the complexity dichotomy into quasi-polynomial and NP-complete was fully achieved for the class of list partitions into four parts. A quasi-polynomial algorithm for LIST SKEW PARTITION was presented, an indication that the problem would be in P.

Many combinatorial problems can be described as finding a partition of the vertices of a given graph into subsets satisfying certain properties *internally* (some parts may be required to be stable sets, others may conversely be required

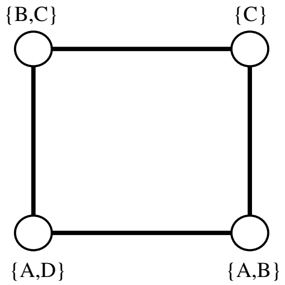

Fig. 2. Instance G, L admits a list skew partition.

a
$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & * & 1 & * & * \\ B & 1 & * & * & * \\ C & * & * & * & 0 \\ D & * & * & 0 & * \end{array}$$

b
$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & * & 1 & * & * \\ B & 1 & * & * & * \\ C & * & * & * & 1 \\ D & * & * & 1 & * \end{array}$$

c
$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 1 & * & * & * \\ B & * & 0 & * & * \\ C & * & * & 0 & 0 \\ D & * & * & 0 & * \end{array}$$

Fig. 3. Matrices for the SKEW (a), $2K_2$ (b), and STUBBORN (c) partition problems.

to be complete sets), and *externally* (some pairs of parts may be required to be completely nonadjacent, others completely adjacent). Following the notation introduced by Feder, Hell, Klein, and Motwani [23], the basic family of partition problems considered is known as *M-partition*: partition the vertex set of a graph into k parts, $A_1, A_2, \ldots, A_k$, with a fixed "pattern" of requirements as to which A_i are stable or complete and which pairs A_i, A_j are completely nonadjacent or completely adjacent. These requirements may be conveniently encoded by a corresponding symmetric $k \times k$ matrix M: the diagonal entry M_{ii} is 0 if A_i is required to be a stable set, 1 if A_i is required to be a clique, and $*$ otherwise (no restriction); the off-diagonal entry M_{ij} is 0, 1, or $*$, if A_i and A_j are required to be completely nonadjacent, completely adjacent, or to have arbitrary connections, respectively.

Many combinatorial problems just ask for an M-partition. For instance, a k-colouring is an M-partition in which the k diagonal entries of M are all 0. Other well-known problems ask for M-partitions in which all parts are restricted to be nonempty (e.g., skew partition, star cutset, clique cutset, stable cutset, 1-join, 2-join). In yet other problems there are additional constraints, such as those in the definition of a homogeneous set (requiring one of the parts to have at least two and at most $n-1$ vertices). The most convenient way to express these additional constraints turns out to be to allow specifying for each vertex (as part of the input) a "list" of parts in which the vertex is allowed to be. Specifically, the LIST M-PARTITION problem asks for an M-partition of the input graph in which each vertex is placed in a part that is in its list. Both the basic M-PARTITION problem ("Does the input graph admit an M-partition?") and the problem of existence of an M-partition with all parts nonempty admit polynomial-time reductions to the LIST M-PARTITION problem as do all of the above problems with the "additional" constraints. List partitions generalize list-colourings, which have proved very fruitful in the study of graph colourings.

Feder et al. [23] were the first to introduce and investigate the list version of these problems. It turned out to be a useful generalization, since list problems recurse more conveniently. They classified the complexity (as polynomial-time solvable or NP-complete) of list M-partition problems for all 3×3 matrices M and some 4×4 matrices M. For other 4×4 matrices M they were able to produce sub-exponential algorithms—including one for the skew partition problem. This was the first sub-exponential algorithm for the problem, and an indication that the problem was not likely to be NP-complete. Subsequently, and motivated by their approach, we were able to show that in fact one can use the mechanism of list partitions to give a *polynomial-time* algorithm for the problem [26].

Our recursive algorithm for LIST SKEW PARTITION recursively defines simpler list skew partition problems with smaller simpler lists. The number of subproblems $T(n)$ encountered during recursive skew partitioning satisfies nested recurrences of the form $T(n) \leq 4\,T(9n/10)$, which yield the running time $O(n^{100})$, a challenge to the accepted notion that polynomial-time solvable is the same as efficiently solvable in practice.

Every LIST M-PARTITION problem with M of dimension 4 was classified by Feder et al. [23] as either solvable in quasi-polynomial time or NP-complete. Quasi-polynomial time is complexity of $O(n^{c \log^t n})$, where t and c are positive constants and n is the number of vertices in the input graph. In particular, please refer to Fig. 3: LIST M-PARTITION, where M is the skew-partition matrix, was classified as quasi-polynomial-time solvable; LIST M-PARTITION, where M is the $2K_2$-partition matrix, was classified as NP-complete; Cameron et al. [11] showed that all the quasi-polynomial-time cases of the Feder et al. [23] quasi-dichotomy result are actually polynomial-time solvable, with the sole exception of the LIST STUBBORN PARTITION problem, where in the stubborn problem the only external constraint is that every vertex of part C is nonadjacent to every vertex of part D, the internal constraints are that part C and part B are required to be stable sets, and that part A is required to be a clique. Kennedy and Reed [39] announced recently a more efficient $O(n^6)$-time algorithm for SKEW PARTITION,

Table 1
N: NP-complete, P: polynomial, Q: quasi-polynomial, O: open.

Problem	List	Singleton list	Nonempty part
SKEW	n^{100}	n^{100}	n^6
$2K_2$	N	N	O
STUBBORN	Q	P	P

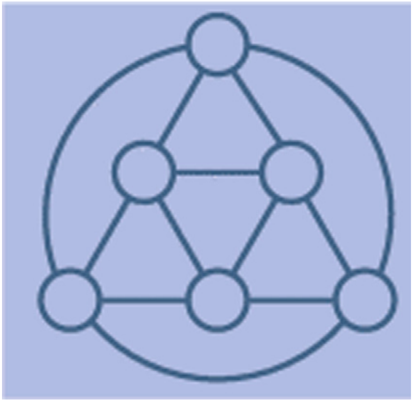

Fig. 4. The Lagos graph.

in which they did not consider the more general LIST SKEW PARTITION problem. Dantas et al. [22] studied *H-partitions*, where the matrix M has dimension 4 and only $*$s on the main diagonal (i.e., no internal constraints are imposed) and all parts must be nonempty. All such nonempty part partitions with external constraint problems have been shown to be polynomial-time solvable [11,22,23,26], except for $2K_2$-PARTITION [19].

The usual polynomial reduction from nonempty part partition to list partition considers the solution of $O(n^4)$ particular singleton list problems such that four special vertices x_A, x_B, x_C, x_D have singleton lists A, B, C, D, respectively, and the remaining $n-4$ vertices have lists $ABCD$. This reduction was used in [26] to reduce SKEW PARTITION, a nonempty part partition problem, to LIST SKEW PARTITION. The particular SINGLETON LIST STUBBORN PARTITION problem such that all but four special vertices x_A, x_B, x_C, x_D have lists $ABCD$ can be solved [22] by noticing that a vertex containing C in its list must also contain D, which gives a refined set of nonsingleton lists: $\{ABCD, ABD, ACD, BCD, AB, AD, BD, CD\}$. Therefore, we may place all vertices containing D in their nonsingleton lists into D and check whether the remaining graph (subgraph induced by the vertices with list AB) is a split graph, which can be done by applying 2-SAT, yielding an $O(n^2)$-time algorithm [3]. Therefore, SINGLETON LIST STUBBORN PARTITION is in P, which implies that NONEMPTY PART STUBBORN PARTITION is in P. On the other hand, and perhaps surprisingly, we may reduce LIST $2K_2$-PARTITION to SINGLETON LIST $2K_2$-PARTITION, thus establishing that SINGLETON LIST $2K_2$-PARTITION is NP-complete[2]: from an instance G, L of LIST $2K_2$-PARTITION construct an instance G', L' of SINGLETON LIST $2K_2$-PARTITION by setting G' to be graph G with additional four vertices x_A, x_B, x_C, x_D, such that $N'(x_A)$ contains x_B and all vertices z satisfying $B \in L(z)$, and similarly for vertices x_B, x_C, x_D. As a consequence, a possible polynomial algorithm for NONEMPTY PART $2K_2$-PARTITION cannot use the usual strategy of considering the SINGLETON LIST $2K_2$-PARTITION problem, or maybe the reduction indicates that indeed NONEMPTY PART $2K_2$-PARTITION is NP-complete.

The above discussion, summarized in Table 1, suggests a complexity-separating question.

Is LIST PARTITION *harder than* NONEMPTY PART PARTITION?

2.2. Clique graph

A *complete set* of a graph $H = (V, E)$ is a subset of V inducing a complete subgraph. A *clique* is a maximal complete set. The *clique family* of H is denoted by $\mathcal{C}(H)$. The *clique graph* of H is the intersection graph of $\mathcal{C}(H)$. The *clique operator* assigns to each graph H its clique graph, which is denoted by $K(H)$. We say that a graph G is *a clique graph* if G belongs to the image of the clique operator, i.e., if there exists a graph H such that $G = K(H)$.

Note that the number of maximal complete sets may be exponential on the number of vertices. Consider, for instance, the graph consisting of an induced matching. It contains $2^{n/2}$ maximal stable sets; thus its complement is a graph with $2^{n/2}$ maximal complete sets. The graph in the Lagos 2009 logo depicted in Fig. 4 is the complement of an induced matching of size 3, and it has eight cliques.

2 This argument is due to Fabio Protti, private communication.

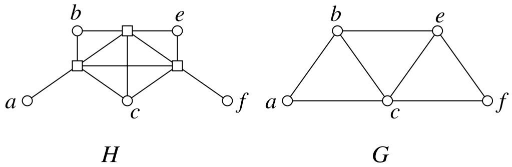

Fig. 5. G is the clique graph of H.

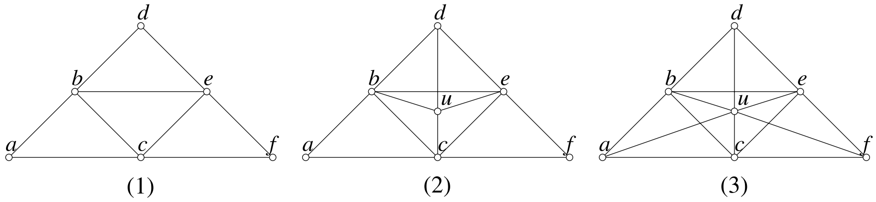

Fig. 6. (1) clique-complete, but non-clique-Helly graph, non-clique graph; (2) clique-complete, non-clique-Helly graph, but clique graph; (3) clique-complete, clique-Helly graph, clique graph.

Consider the time complexity of the problem of recognizing clique graphs. This is the time complexity of the following decision problem.

CLIQUE GRAPH
Instance: Graph G.
Question: Is there a graph H such that graph G is the intersection graph of the cliques of graph H?

The Helly property has been much studied with the goal of classifying the CLIQUE GRAPH problem. A family of sets $\mathcal{F} = (F_i)_{i \in I}$ is *pairwise intersecting* if the intersection of any two members is not the empty set. The *total intersection* of $\mathcal{F}$ is the set $\bigcap \mathcal{F} = \bigcap_{i \in I} F_i$. The family $\mathcal{F}$ has the *Helly property* if any pairwise intersecting subfamily has nonempty total intersection. The complete set C *covers* the edge uv when u and v belong to set C. A *complete set edge-cover* of a graph G is a family of complete sets of G covering all edges of G.

Roberts and Spencer [56] proved the following characterization: G is a clique graph if and only if G admits an edge-cover by complete sets satisfying the Helly property. The characterization by such a special edge-cover by complete sets – known as the RS-family – leads to a proof that CLIQUE GRAPH is in NP. An RS-family of size at most $|E(G)|$ gives the desired graph H such that $|V(H)| \leq |V(G)| + |E(G)|$. In Fig. 5, H was constructed after the RS-family of G consisting of the three triangles of G. Note that H has five cliques and eight vertices. Each of the three square vertices of graph H corresponds to a triangle of G, and the remaining five vertices of H are labelled so that they correspond to the five vertices of G. Each vertex of G corresponds to a clique of H, and G is indeed the clique graph of H.

Notice that for any graph G the clique family $\mathcal{C}(G)$ is a complete set edge-cover of G, but, in general, this family does not satisfy the Helly property. Graphs such that $\mathcal{C}(G)$ satisfies the Helly property are called *clique-Helly* graphs [34]. The characterization by Roberts and Spencer [56] implies that every clique-Helly graph is a clique graph. The converse is not true: there exist clique graphs which are not clique-Helly graphs. Fig. 6 shows three examples: (1) a non-clique graph (no complete set edge-cover satisfies the Helly property [56]); (2) a clique graph that is not a clique-Helly graph (the clique family does not satisfy the Helly property, but the complete set edge-cover $\{a, b, c\}$, $\{c, e, f\}$, $\{b, d, u\}$, $\{d, e, u\}$, $\{b, c, e, u\}$ does); and (3) a clique graph that is a clique-Helly graph (the clique family has the Helly property). These examples also show that being a clique graph or being a clique-Helly graph is not a hereditary property.

Hamelink [34] defined clique-Helly graphs and proved that every clique-Helly graph is a clique graph. The concept of the RS-family generalized the proof of Hamelink from an edge-cover by cliques to an edge-cover by complete sets. Szwarcfiter [63] subsequently gave a polynomial-time algorithm to recognize clique-Helly graphs. Mello, Lucchesi, and Szwarcfiter [44] considered *clique-complete* graphs (graphs whose clique family is mutually intersecting) and proved that the corresponding recognition problem is coNP-complete. It is remarkable that to test whether the clique family of a given graph is mutually intersecting is coNP-complete, whereas to test whether the clique family satisfies the Helly property is polynomial. Fig. 6 shows three graphs such that all of them have four cliques and all of them are clique complete. The graph of Fig. 6(1) is the smallest non-clique graph.

The NP-completeness of the CLIQUE GRAPH problem was obtained in collaboration with Liliana Alcón, Luerbio Faria, and Marisa Gutierrez [2]. The chosen NP-complete problem was a variation of 3SAT [30,47].

3SAT$_{\overline{3}}$
Instance: $I = (U, C)$, where $U = \{u_i : 1 \leq i \leq n\}$ is a set of Boolean variables, and $C = \{c_j : 1 \leq j \leq m\}$ a set of clauses over U such that each clause has two or three variables, and each variable occurs at most three times in C.
Question: Is there a truth assignment for U such that each clause in C has at least one true literal?

Let $I = (U, C)$ be any instance of 3SAT$_{\overline{3}}$. For the NP-completeness proof, it is convenient to assume with no loss of generality that each variable occurs two or three times in C, and no variable occurs twice in the same clause. In addition,

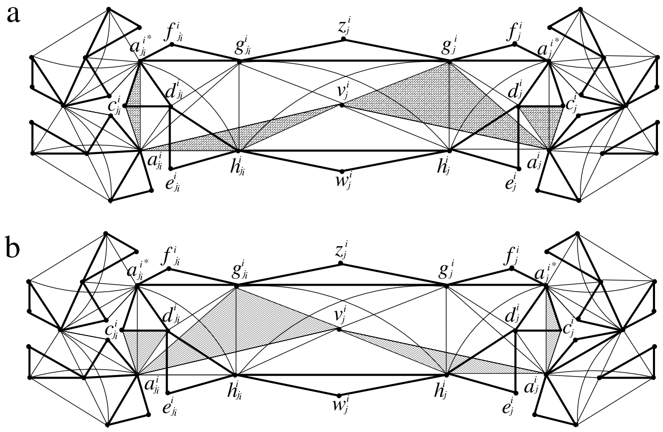

Fig. 7. For any variable u_i and $j \in \bar{J}_i$, any RS-family of G_I must contain either the filled triangles in (a) or the filled triangles in (b). In any case the bold triangles must belong to the RS-family.

Table 2
N: NP-complete, P: polynomial, O: open.

Graph class	VERTEXCOL	EDGECOL	MAXCUT
Perfect	P	N	N
Chordal	P	O	N
Split	P	O	N
Strongly chordal	P	O	O
Comparability	P	N	O
Bipartite	P	P	P
Permutation	P	O	O
Cographs	P	O	P
Proper interval	P	O	O
Split-proper interval	P	P	P

if u_i occurs twice in C, then we assume that it occurs once as literal u_i and once as literal $\overline{u_i}$; and if u_i occurs three times in C, then we assume that it occurs once as literal u_i and twice as literal $\overline{u_i}$. These assumptions allow the following convenient notation: for each variable u_i, let j_i be the subindex of the unique clause where variable u_i occurs as literal u_i; and $\bar{J}_i = \{j : \text{ literal } \overline{u_i} \text{ occurs in } c_j\}$.

Fig. 7 exhibits the variable gadget corresponding to variable u_i, a subgraph of the constructed graph G_I. Note in the variable gadget several bold triangles that are cliques, a property that forces a triangle to be present in every RS-family. The key property of the variable gadget is that every RS-family of G_I must contain either the filled triangles in (a), called the *false* triangles, or the filled triangles in (b), called the *true* triangles. All bold triangles must belong to the RS-family.

Note that the smallest non-clique graph depicted in Fig. 6(1) is a planar split graph. The NP-completeness of CLIQUE GRAPH suggests the study of the problem restricted to classes of graphs not properly contained in the class of clique graphs.

Is CLIQUE GRAPH *polynomial for split graph instances?*

The next section discusses the role of split graphs in the context of complexity-separating problems and complexity-separating graph classes.

3. NP-completeness ongoing guide

In his famous NP-completeness columns [37], Johnson updates his seminal book, Computers and Intractability [30]. We refer to the column on graph restrictions and their effect [38]. The goal is to identify interesting problems and interesting graph classes establishing the concept of complexity-separating questions. A *complexity-separating graph class* $\mathcal{C}$ separates two problems π and σ if π is NP-complete when restricted to $\mathcal{C}$-graph inputs but σ is P when restricted to $\mathcal{C}$-graph inputs. A *complexity-separating problem* π separates two graph classes $\mathcal{A} \subset \mathcal{B}$ if π is P when restricted to $\mathcal{A}$-graph inputs but π is NP-complete when restricted to $\mathcal{B}$ inputs. The column has sections devoted to perfect graphs and to intersection graphs.

Table 2 is a subtable of the table presented in [38] from where we have selected some rows (graph classes) and some columns (problems), and we have kept the same notation.

It is well known that perfect graphs constitute a graph class for which VERTEX COLOURING is polynomial. On the other hand, EDGE COLOURING is NP-complete for perfect graph inputs. Actually, the subclass of comparability graphs is complexity

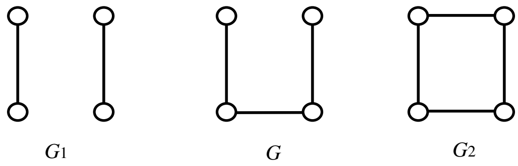

Fig. 8. G is a sandwich split graph for pair G_1, G_2.

separating [36] because it separates problems VERTEX COLOURING and EDGE COLOURING. In fact, EDGE COLOURING is itself a complexity-separating problem that separates comparability graphs and bipartite graphs. It is remarkable that there are very few graph classes for which the complexity of EDGE COLOURING is established, and it is surprising that, since 1985, several graph classes for which Johnson stated EDGE COLOURING as "apparently open, but possibly easy to resolve" remain stubbornly open. The class of cographs, a very structured graph class defined by forbidding the path on four vertices P_4 as an induced subgraph, admits only a partial solution [57]. The graph class of split-proper interval graphs deserves attention. Ortiz, Maculan, and Szwarcfiter [46] characterized graphs that are required to be split and proper interval. By using this characterization, polynomial algorithms have been found for EDGE COLOURING [46] and for MAXCUT [6]. Note that although often stated otherwise [59], MAXCUT is open when restricted to proper interval graphs.

I should point out[3] that there are classes of graphs for which VERTEX COLOURING is NP-complete and EDGE COLOURING is polynomial, for instance graphs with a universal vertex [49]. Perhaps even more interesting is the existence of classes of graphs for which EDGE COLOURING is NP-complete but TOTAL COLOURING (where we proper colour all elements of a graph, vertices and edges) is polynomial [42].

3.1. Split versus chordal

The fact that the graph classes split and chordal agree with respect to VERTEX COLOURING, EDGE COLOURING, and MAXIMUM CUT suggests a pattern that has been the subject of much research.

Split graphs constitute a very structured subclass of chordal graphs, being graphs such that both the graph and its complement are required to be chordal. This strong requirement forces the vertex set of a split graph to admit a partition into a stable set and a clique.

In 1985, Johnson [38] stated "I know of no problem that separates the two classes in complexity". Twenty years later, Spinrad, in his book [59], gives a survey of results on graph classes, an update of the complexity status regarding complexity-separating problems. MAXIMUM CLIQUE, VERTEX COLOURING are in P, actually solvable in linear time, for both chordal and split graphs, whereas DOMINATING SET, MAXCUT, HAMILTON CYCLE are NP-complete for both chordal and split graphs. There are a few complexity-separating problems (for instance TRIANGLE PACKING and PATHWIDTH) for which the problem is NP-complete for chordal but polynomial for split graphs. Spinrad [59] states: "split graphs often are at the core of algorithms and hardness results for chordal graphs".

Complexity-separating problems for chordal and split graphs are rare. The existing literature suggests two possible additional complexity-separating problems: EDGE COLOURING and CLIQUE GRAPH. The partial results for EDGE COLOURING when restricted to chordal graphs, interval graphs, and split graphs indicate the problem is challenging even when restricted to very structured graph classes [24,46]. The recent NP-completeness proof of CLIQUE GRAPH suggests considering the graph class of planar graphs and the graph class of chordal graphs, in particular the class of split graphs [1,2].

3.2. Graph sandwich problem

A graph $G_1 = (V, E_1)$ is a *spanning* subgraph of $G_2 = (V, E_2)$ if $E_1 \subseteq E_2$; a graph $G = (V, E)$ is a *sandwich* graph for the pair G_1, G_2 if $E_1 \subseteq E \subseteq E_2$. For notational simplicity in what follows, we let E_3 be the set of all edges in the complete graph with vertex set V which are not in E_2. Thus every sandwich graph for the pair G_1, G_2 satisfies $E_1 \subseteq E$ and $E \cap E_3 = \emptyset$. We call E_1 the *forced edge set*, $E_2 \setminus E_1$ the *optional edge set*, and E_3 the *forbidden edge set*. Fig. 8 shows an instance and a solution for GRAPH SANDWICH PROBLEM FOR SPLIT GRAPHS.

GRAPH SANDWICH PROBLEM FOR PROPERTY Π
Instance: Two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$.
Question: Is there a graph $G = (V, E)$ with $E_1 \subseteq E \subseteq E_2$ that satisfies property Π?

We shall use both forms (V, E_1, E_2) and (V, E_1, E_3) to refer to an instance of a graph sandwich problem. The recognition problem for a class of graphs $\mathcal{C}$ is equivalent to the graph sandwich problem in which the forced edge set $E_1 = E$, the optional edge set $E_2 \setminus E_1 = \emptyset$, $G = (V, E)$ is the graph we want to recognize, and property Π is "to belong to class $\mathcal{C}$".

In their seminal paper, Golumbic et al. [31] introduced sandwich problems and studied this class of problems with respect to several subclasses of perfect graphs proving that the GRAPH SANDWICH PROBLEM FOR SPLIT GRAPHS remains polynomial.

[3] I thank Flavia Bonomo for this question after my talk!

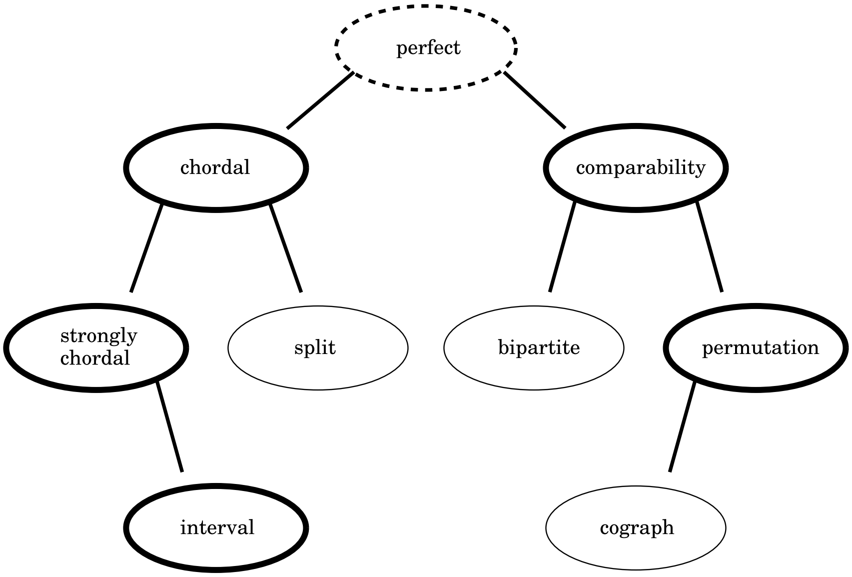

Fig. 9. — NP-complete; — polynomial; — — — open.

On the other hand, they proved that the GRAPH SANDWICH PROBLEM FOR PERMUTATION GRAPHS turns out to be NP-complete. Since a sandwich problem generalizes a recognition problem, the interest is to search for problems for which the recognition is polynomial but the sandwich version is NP-complete. For instance, the 1-JOIN COMPOSITION RECOGNITION PROBLEM is polynomial [20] whereas the 1-JOIN COMPOSITION SANDWICH PROBLEM turns out to be NP-complete [27]. Note that the recognition problem for clique cutset, star cutset and skew cutset are all in P, whereas the sandwich problem presents a non-monotonicity: for clique cutset it is NP-complete, for star cutset it is polynomial, and for skew cutset is NP-complete [65,68].

Polynomial graph sandwich problems are rare. In the seminal paper [31], where this generalization of recognition problems was proposed, several classes of perfect graphs were considered with NP-completeness proofs for the corresponding sandwich problems with only two exceptions: split graphs and cographs, for which the sandwich problems were proved to be polynomial. Further work found two additional examples of polynomial graph sandwich problems also arising in the context of perfect graphs: the HOMOGENEOUS SET SANDWICH PROBLEM [12] and the STAR CUTSET SANDWICH PROBLEM [65].

Fig. 9 updates the diagram by Golumbic et al. [31] that considered with respect to the sandwich problem the original diagram by Johnson [38]. At the time of Johnson [38], the recognition problem for all those graph classes was known to be polynomial, with the exception of the recognition problem for perfect graphs, a famous open problem at that time, classified recently to be polynomial [15]. Although the recognition problem for all those graph classes is polynomial, for several of them the relaxation given by the sandwich problem turns out to be NP-complete. Recently, two open problems proposed in [31] were settled as NP-complete: the GRAPH SANDWICH PROBLEM FOR STRONGLY CHORDAL GRAPHS [28] and the GRAPH SANDWICH PROBLEM FOR CHORDAL BIPARTITE GRAPHS [61]. A challenging open problem is the GRAPH SANDWICH PROBLEM FOR PERFECT GRAPHS. It is the only remaining open problem among those proposed by Golumbic et al. [31]. It is remarkable that the recognition of Berge trigraphs defined by Chudnovsky [16] corresponds to the GRAPH SANDWICH PROBLEM FOR IMPERFECT GRAPHS. It is also interesting that the GRAPH SANDWICH PROBLEM FOR CHORDAL GRAPHS is NP-complete, whereas the GRAPH SANDWICH PROBLEM FOR SPLIT GRAPHS is polynomial [31].

4. Two full dichotomies

We shall discuss two full dichotomies, by presenting two classes of problems that we have been able to define such that for each class every problem is classified into P or NP-complete.

The first class of problems considers sandwich problems [31] and the full dichotomy for the three NONEMPTY PART SANDWICH problem. For every graph partition into three nonempty parts, we were able to classify the corresponding sandwich problem [66]. The second class of problems considers EDGE COLOURING [36]. We were able to identify the graph class of graphs with no cycle with a unique chord [69], a class of graphs such that the complexity of EDGE COLOURING is obtained for every possible value of the maximum vertex degree [41].

4.1. The three nonempty part problem

To motivate the study of the sandwich problem for three nonempty part partition, we need to make some considerations about two-part partitions and about some related classes of problems for which the full dichotomy had been established

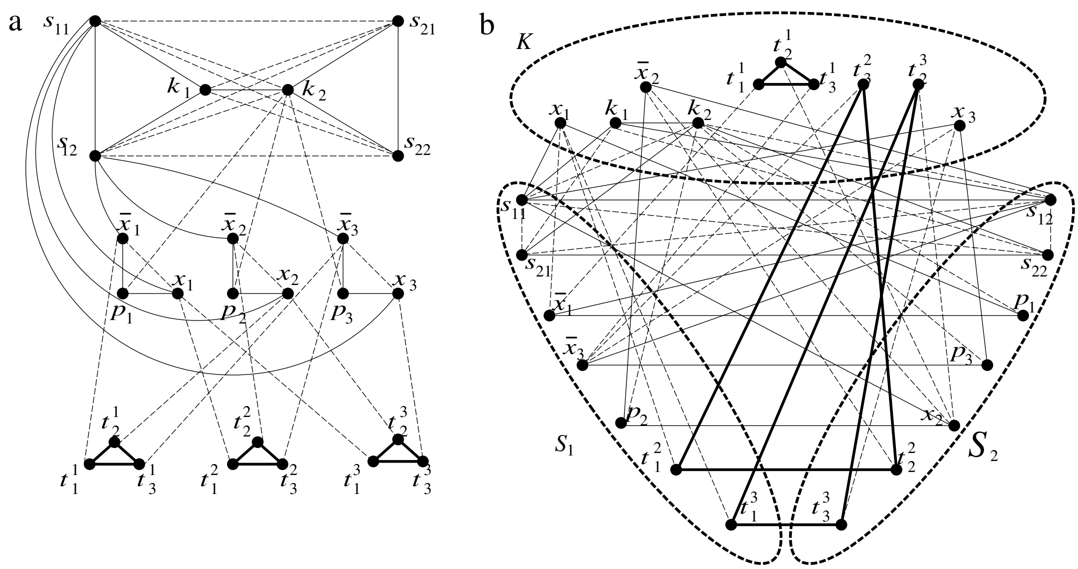

Fig. 10. (a) Instance (V, E_1, E_3) obtained from the satisfiable instance of 3SAT: $I = (U, C) = (\{x_1, x_2, x_3\}, \{(\overline{x_1} \vee x_2 \vee \overline{x_3}), (x_1 \vee \overline{x_2} \vee \overline{x_3}), (x_1 \vee x_2 \vee x_3)\})$ and (b) respective partition for the $(2, 1)$ graph G defined from the satisfying truth assignment $x_1 = F, x_2 = T, x_3 = F$.

previously: the three NONEMPTY PART RECOGNITION problem and the GRAPH RECOGNITION PROBLEM FOR (k, l) GRAPHS—the so-called generalized split graphs.

The GRAPH SANDWICH PROBLEM FOR SPLIT GRAPHS was proved polynomial in the seminal paper of sandwich problems [31] by a reduction to 2-SAT. All remaining two-part sandwich problems can be proved to be in P by a similar reduction, where each vertex corresponds to a variable, and the parts A, B are associated with the values true and false. The forced edge set E_1 and the forbidden edge set E_3 correspond to a set of 2-SAT clauses, in such a way that different two-part problems have different forcing rules [66].

A graph G is (k, l) if its vertex set can be partitioned into at most k stable sets and l cliques. The particular case $k = l = 1$ is the class of split graphs and a (k, l)-graph is called a *generalized split* graph. The full dichotomy for the GRAPH RECOGNITION PROBLEM FOR (k, l) GRAPHS had been completely determined as follows [7]: if $k = 3$ and $l = 0$ then the corresponding problem is 3-colouring, which implies that the recognition of (k, l) graphs is NP-complete, whenever $k \geq 3$ or $l \geq 3$. For the remaining values of k and l, the problem is polynomial: $(1, 1)$ graphs are split graphs; $(2, 0)$ graphs are the bipartite graphs; and the polynomial-time recognition of $(2, 1)$ graphs and of $(2, 2)$ graphs had been established in [7,8,23]. By proving that GRAPH SANDWICH PROBLEM FOR (k, l) GRAPHS is NP-complete for the cases $(2, 1)$ and $(2, 2)$, the full dichotomy for GRAPH SANDWICH PROBLEM FOR (k, l) GRAPHS was completely determined [21]. The problem is NP-complete if $k+l > 2$; the problem is polynomial otherwise. Consider in Fig. 10 the constructed instance for the GRAPH SANDWICH PROBLEM FOR $(2, 1)$ GRAPHS.

The full dichotomy for the three NONEMPTY PART RECOGNITION problem had been completely determined [23]: apart from 3-COLOURING, the only such recognition problem classified as NP-complete is STABLE CUTSET [9,40].

Regarding sandwich problems, two interesting cases of partitions into three nonempty parts had been classified: the HOMOGENEOUS SET SANDWICH PROBLEM as polynomial [12] and the CLIQUE CUTSET SANDWICH PROBLEM as NP-complete [65].

The next natural step was to establish the full dichotomy for the three NONEMPTY PART SANDWICH problem. If all entries of a matrix M are 0 or $*$, then M defines a hereditary property, and the sandwich problem is a recognition problem, for which it is sufficient to test whether G_1 admits a three nonempty part M-partition. If all entries of a matrix M are 1 or $*$, then M defines an ancestral property, and the sandwich problem is a recognition problem, for which it is sufficient to test whether G_2 admits a three nonempty part M-partition. Since all three NONEMPTY PART M-PARTITION RECOGNITION problems were classified, we focused on *interesting* matrices containing at least one entry 0 and one entry 1, which gave a challenging task of classification of 61 interesting problems, among them 19 classified as NP-complete and 42 as polynomial [66]. For instance, the addition of distinct internal constraints to the homogeneous set problem provided three nonempty part sandwich problems that are polynomial or NP-complete.

Subsequently, the EXTERNAL CONSTRAINT FOUR NONEMPTY PART SANDWICH problem was considered [67,68] with the goal of further studying the challenging partition problems depicted in Fig. 11. Note that skew partition, 1-join composition, and $2K_2$-partition are external constraint four nonempty part partitions. For both skew partition and 1-join composition, the same dichotomy holds: recognition is in P whereas the sandwich problem is NP-complete [27,68]. Note that $2K_2$-partition defines an ancestral property, and so the recognition and the sandwich problem must have the same complexity. Additionally, we were able to define the class of $2K_2$-hard problems, containing several EXTERNAL CONSTRAINT FOUR NONEMPTY PART SANDWICH problems. The classification obtained for the EXTERNAL CONSTRAINT FOUR NONEMPTY PART SANDWICH was into NP-complete, polynomial, or $2K_2$-hard [67].

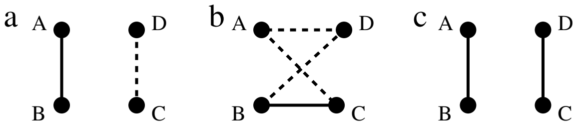

Fig. 11. (a) Skew partition, (b) 1-join composition, and (c) $2K_2$-partition.

Table 3
Complexity dichotomy for edge-colouring graphs with no cycle with unique chord.

Graph class	$\Delta = 3$	$\Delta \geq 4$	Regular
Graphs of $\mathcal{C}$	NP-complete	NP-complete	NP-complete
4-hole-free graphs of $\mathcal{C}$	NP-complete	Polynomial	Polynomial
6-hole-free graphs of $\mathcal{C}$	NP-complete	NP-complete	NP-complete
{4-hole, 6-hole}-free graphs of $\mathcal{C}$	Polynomial	Polynomial	Polynomial

Table 4
Complexity dichotomy for edge-colouring multipartite graphs.

Graph class	$k \leq 2$	$k \geq 3$
k-partite graphs	Polynomial	NP-complete

4.2. Graphs with no cycle with a unique chord

Trotignon and Vušković [69] studied the class $\mathcal{C}$ of graphs that do not contain a cycle with a unique chord. The main motivation for investigating this class was to find a structure theorem for it, a kind of result that is not very frequent in the literature. Basically, a structure result states that every graph in $\mathcal{C}$ can be built starting from a restricted set of basic graphs and applying a series of known "gluing" operations. Another interesting property of this class is that it belongs to the family of the χ-bounded graphs, introduced by Gyárfás [33] as a natural extension of perfect graphs. A family of graphs $\mathcal{G}$ is χ*-bounded* with χ*-binding function* f if, for every induced subgraph G' of $G \in \mathcal{G}$, we have $\chi(G') \leq f(\omega(G'))$, where $\chi(G')$ denotes the chromatic number of G' and $\omega(G')$ denotes the size of a maximum clique in G'. The research in this area is mainly devoted to understanding for what choices of forbidden induced subgraphs the resulting family of graphs is χ-bounded; see [54] for a survey. Note that perfect graphs are a χ-bounded family of graphs with χ-binding function $f(x) = x$, and perfect graphs are characterized by excluding odd holes and their complements, where a *hole* is a chordless cycle of length at least 4. Also, by Vizing's Theorem, the class of line graphs of simple graphs is a χ-bounded family with χ-binding function $f(x) = x + 1$ (this special upper bound is known as the *Vizing bound*) and line graphs are characterized by nine forbidden induced subgraphs [70]. The class $\mathcal{C}$ is also χ-bounded with the Vizing bound [69]. Also in [69] the following results are obtained for graphs in $\mathcal{C}$: an $O(nm)$ algorithm for VERTEX COLOURING, an $O(n + m)$ algorithm for MAXIMUM CLIQUE, an $O(nm)$ recognition algorithm, and the NP-completeness of MAXIMUM STABLE SET.

We have considered the complexity of the edge-colouring problem in $\mathcal{C}$ [41]. The edge-colouring problem or the chromatic index problem is the problem of determining the least number $\chi'(G)$ of colours needed in an edge-colouring of G. We have also investigated the subclasses obtained from $\mathcal{C}$ by forbidding 4-holes and/or 6-holes. Tables 3 and 4 summarize the main results, showing that, even for graph classes with strong structure, the edge-colouring problem may be difficult. We denote by $\mathcal{C}$ the class of graphs that do not contain a cycle with a unique chord and by Δ the maximum degree.

The class initially investigated in [41] was the class $\mathcal{C}$ of graphs with no cycle with a unique chord. For the purposes of that work, a graph G is *basic* if G is a complete graph, a hole with at least five vertices, a strongly 2-bipartite graph, or an induced subgraph of the Petersen graph or of the Heawood graph; and G has no 1-cutset, proper 2-cutset or proper 1-join. We have proved that edge-colouring is NP-complete for graphs in $\mathcal{C}$. We have considered, then, a subclass $\mathcal{C}' \subset \mathcal{C}$ whose graphs are the graphs in $\mathcal{C}$ that do not have a 4-hole. By forbidding 4-holes we avoid decompositions by proper 1-joins, which are difficult to deal with in edge-colouring [4,57,58]. That is, each non-basic graph in $\mathcal{C}'$ can be decomposed by 1-cutsets and proper 2-cutsets. For this class $\mathcal{C}'$ we have established a dichotomy: edge-colouring is NP-complete for graphs in $\mathcal{C}'$ with maximum degree 3 and polynomial for graphs in $\mathcal{C}'$ with maximum degree *not* 3. We have also determined a necessary condition for a graph $G \in \mathcal{C}'$ of maximum degree 3 to be Class 2, satisfying chromatic index $\chi'(G) = \Delta(G) + 1 = 4$. This condition is having graph P^* – a subgraph of the Petersen graph – as a basic block in the decomposition tree. As a consequence, if both 4-holes and 6-holes are forbidden, the chromatic index of graphs with no cycle with unique chord can be determined in polynomial time. The results achieved in [41] have connections with other areas of research in edge-colouring, as we describe in the following four observations.

The first observation refers to the complexity dichotomy result found for class $\mathcal{C}'$. This dichotomy presents great interest since, to the best of our knowledge, this is the first class for which edge-colouring is NP-complete for graphs with a given fixed maximum degree Δ but is polynomial for graphs with maximum degree $\Delta' > \Delta$. Moreover, Class $\mathcal{C}'$ is the first interesting graph class for which edge-colouring is NP-complete in general, but is polynomial when restricted to regular

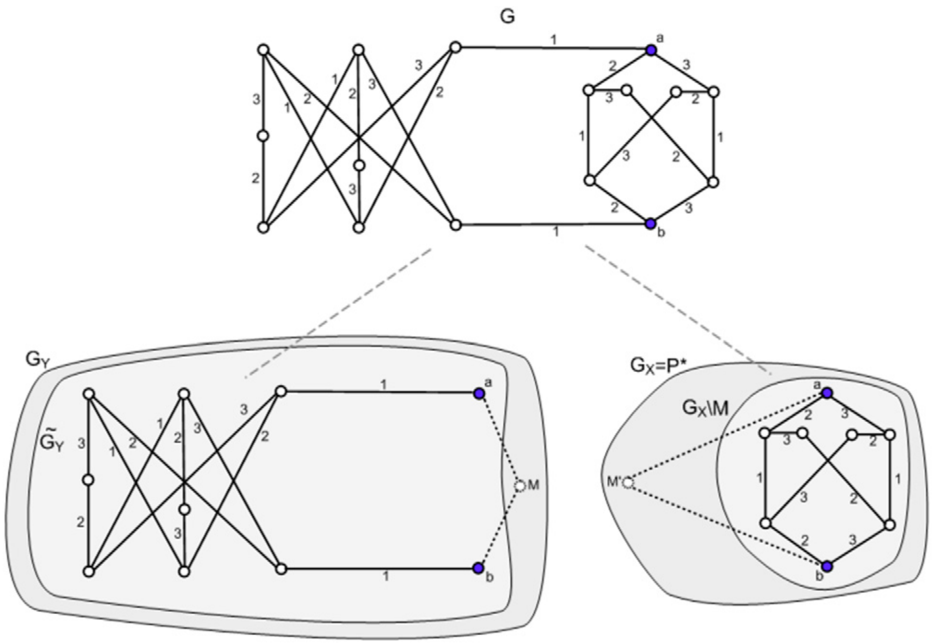

Fig. 12. Combining edge-colourings with respect to 2-cutset.

graphs. It is interesting to observe that $\mathcal{C}'$ is a graph class with few regular graphs—only the Petersen graph, the Heawood graph, the complete graphs and the holes.

The second observation refers to a conjecture of Chetwynd and Hilton. An important tool to identify Class 2 graphs is the concept of *overfullness* [25]. A graph $G = (V, E)$ is *overfull* if $|E| > \Delta(G)\lfloor |V|/2 \rfloor$; G is *subgraph overfull* if it has a subgraph of same maximum degree that is overfull. Subgraph-overfull graphs are Class 2 [25], and it can be verified in polynomial time whether a graph is subgraph overfull [45]. For some graph classes, being subgraph overfull is equivalent to being Class 2. Examples of such classes are graphs with a universal vertex [49], complete multipartite graphs [35], and split graphs with odd maximum degree [13]. A conjecture of Chetwynd and Hilton [14] states that a graph $G = (V, E)$ with $\Delta(G) > |V|/3$ is Class 2 if and only if is subgraph overfull. In fact, for most graph classes for which the edge-colouring problem can be solved in polynomial time, the equivalence "Class 2 = Subgraph Overfull" holds. It is remarkable that the majority of these classes are composed of graphs whose maximum degree is large, always larger than one third of the number of vertices. So, for these graph classes, the equivalence "Class 2 = Subgraph Overfull" – and the consequent polynomial-time algorithm for the edge-colouring problem – would be a direct consequence of the Subgraph Overfull Conjecture, in the case of its validity. In this sense, the class $\mathcal{C}'$ investigated in [41,69] presents great interest: for graphs in $\mathcal{C}'$ there is no bound on the relation "number of vertices over maximum degree"; yet, if the maximum degree is not 3, the equivalence "Class 2 = Subgraph Overfull" holds. So, the class of the graphs in $\mathcal{C}'$ with maximum degree not 3 is a class of graphs that do not fit the assumptions of Subgraph Overfull Conjecture, but for which edge-colouring is still solvable in polynomial time through the equivalence "Class 2 = Subgraph Overfull".

The third observation is related to the study of *snarks* [60]. A *snark* is a cubic bridgeless graph with chromatic index 4. In order to avoid trivial cases, snarks are commonly restricted to have girth 5 or more and not to contain three edges whose deletion results in a disconnected graph, each of whose components is non-trivial. The study of snarks is closely related to the Four Colour Theorem. By the result of [41], the only non-trivial snark which has *no* cycle with unique chord is the Petersen graph.

The fourth observation refers to the problem of determining the chromatic index of a k-partite graph, that is, a graph whose vertices can be partitioned into k stable sets. The problem was known to be polynomial for $k = 2$ [37,38] and for complete multipartite graphs [35]. The NP-completeness proof for the chromatic index of graphs in $\mathcal{C}$ implies that edge-colouring is NP-complete for k-partite r-regular graphs, for each $k \geq 3$, $r \geq 3$ [41].

Fig. 12 shows a decomposition with respect to a proper 2-cutset $\{a, b\}$. Note that G is Class 1, a graph for which Δ colours suffice to colour its edges, whereas $G_X = P^*$ is Class 2, any edge-colouring requiring $\Delta + 1$ colours.

5. Proposed complexity-separating questions

Throughout this paper, we have proposed several complexity-separating questions. We conclude by focusing on three main such questions.

Is LIST PARTITION *harder than* NONEMPTY PART PARTITION?

In Section 2.1, we have summarized in Table 1 the complexity status according to classification into NP-complete, Polynomial, Quasi-polynomial, and Open of partition problems into four parts, considering the list and nonempty part versions. There is a large gap between the known polynomial algorithms for LIST SKEW PARTITION and SKEW PARTITION. The LIST STUBBORN problem is the only LIST M-PARTITION problem of dimension 4 of the Quasi dichotomy of Feder et al. [23] classified as Quasi-polynomial that resists classification as NP-complete or polynomial. All partition problems of dimension 4 into nonempty parts and with no internal constrains are polynomial-time solvable with the exception of $2K_2$-PARTITION. The LIST $2K_2$-PARTITION has been classified by Feder et al. [23] as NP-complete. It is remarkable that both classifications of $2K_2$-PARTITION into polynomial or NP-complete are complexity separating. If $2K_2$-PARTITION is classified as NP-complete, it will be the only partition problem of dimension 4 into nonempty parts and with no internal constrains classified as NP-complete. On the other hand, if $2K_2$-PARTITION is classified as polynomial, it will separate the list from the nonempty part versions for partition problems.

Is CLIQUE GRAPH *polynomial for split graph instances*?

In Section 2.2, we discussed the NP-completeness proof that settled the complexity of CLIQUE GRAPH after 40 years. Once this challenging problem is classified as NP-complete, the new challenge now is to further understand which feature of the problem makes it NP-complete. Being a graph theory problem, the natural direction of research is to look for a graph class for which the problem remains NP-complete or turns out to be polynomial. The instance of CLIQUE GRAPH used to prove NP-completeness has clique size $\omega = 12$ and maximum degree $\Delta = 14$, so the problem remains NP-complete even for graphs with bounded clique size ω, and for graphs with bounded maximum degree Δ. However, the problem is polynomial when restricted to graphs with clique size $\omega < 4$ and also when restricted to graphs with maximum degree $\Delta < 5$. Theorem 3 of the fundamental paper by Roberts and Spencer [56] says: a K_4-free graph is a clique graph if and only if it is clique-Helly. A graph with $\Delta < 5$ is a clique graph if and only if it is hereditary clique-Helly [32,50]. This suggests the search of the maximum values for the clique size $3 \leq \omega \leq 11$ and for the maximum degree $4 \leq \Delta \leq 13$ for which the problem is polynomial.

The only reference for the problem of recognizing clique graphs restricted to greater bounded maximum clique size graphs is the class of planar graphs [1]. In that paper, a non-bounded degree subclass of planar clique graphs, larger than clique-Helly planar graphs, and admitting cliques of size 4, is characterized. In addition, a polynomial-time algorithm for the recognition of that subclass of planar clique graphs is given.

Several subclasses of clique graphs have been studied for which polynomial-time recognition is known. In particular, for several classes of graphs the corresponding class of clique graphs is known [64]. Note that it is known that the clique graph of a chordal graph is a dually chordal graph, but the complexity of deciding whether a chordal graph is a clique graph is not known.

Note that the smallest non-clique graph depicted in Fig. 6(1) is a planar split graph. The NP-completeness of CLIQUE GRAPH suggests the study of the problem restricted to classes of graphs not properly contained in the class of clique graphs. The recognition of planar clique graphs and of chordal clique graphs are suggested open problems. The discussion of Section 3.1 suggests CLIQUE GRAPH as a possible complexity-separating problem for the classes of chordal and split graphs.

Is Class $2 =$ *subgraph overfull for chordal graphs*?

In Section 4.2, we discussed the concept of *overfullness* as an important tool to identify Class 2 graphs. In fact, for most graph classes for which the edge-colouring problem can be solved in polynomial time, the equivalence "Class 2 = Subgraph Overfull" holds. We have conjectured that every Class 2 chordal graph is subgraph overfull [25]. The validity of this conjecture implies that the edge-colouring of chordal graphs can be solved in polynomial time. Partial evidence for the subclasses of split graphs [13] and of interval graphs [24] have been obtained.

It is remarkable that, for most graph classes for which EDGE COLOURING can be solved in polynomial time, there is a bound on the "number of vertices over maximum degree": the maximum degree is larger than one third of the number of vertices in order to fit a conjecture of Chetwynd and Hilton [14]. In Section 4.2, we presented the class of {4-hole, cycle with a unique chord}-free graphs with maximum degree not 3 as a class of graphs with no such bound and yet Class 2 = Subgraph Overfull for those graphs. The class of chordal graphs is another graph class with no such bound for which we conjecture Class 2 = Subgraph Overfull.

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