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Hierarchical complexity of 2-clique-colouring weakly chordal graphs and perfect graphs having cliques of size at least 3 [☆]

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ABSTRACT

A clique of a graph is a maximal set of vertices of size at least 2 that induces a complete graph. A k-clique-colouring of a graph is a colouring of the vertices with at most k colours such that no clique is monochromatic. Défossez proved that the 2-clique-colouring of perfect graphs is a Σ_2^P-complete problem (Défossez (2009) [4]). We strengthen this result by showing that it is still Σ_2^P-complete for weakly chordal graphs. We then determine a hierarchy of nested subclasses of weakly chordal graphs whereby each graph class is in a distinct complexity class, namely Σ_2^P-complete, $\mathcal{NP}$-complete, and $\mathcal{P}$. We solve an open problem posed by Kratochvíl and Tuza to determine the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3 (Kratochvíl and Tuza (2002) [7]), proving that it is a Σ_2^P-complete problem. We then determine a hierarchy of nested subclasses of perfect graphs with all cliques having size at least 3 whereby each graph class is in a distinct complexity class, namely Σ_2^P-complete, $\mathcal{NP}$-complete, and $\mathcal{P}$.

1. Introduction

Let $G = (V, E)$ be a simple graph with $n = |V|$ vertices and $m = |E|$ edges. A *clique* of G is a maximal set of vertices of size at least 2 that induces a complete graph. A *k-clique-colouring* of a graph is a colouring of the vertices with at most k colours such that no clique is monochromatic. Any undefined notation concerning complexity classes follows that of Marx [9].

A *cycle* is a sequence of vertices starting and ending at the same vertex, with each two consecutive vertices in the sequence adjacent to each other in the graph. A *chord* of a cycle is an edge joining two nodes that are not consecutive in the cycle.

The *clique-number* $\omega(G)$ of a graph G is the number of vertices of a clique with the largest possible size in G. A *perfect graph* is a graph in which every induced subgraph H needs exactly $\omega(H)$ colours in its vertices so that no K_2 (not necessarily clique) is monochromatic. The celebrated *Strong Perfect Graph Theorem* of Chudnovsky et al. [3] says that a graph is perfect if neither it nor its complement contains a chordless cycle with an odd number of vertices greater than 4. A graph is *chordal* if it does not contain a chordless cycle with a number of vertices greater than 3, and a graph is *weakly chordal* if neither it nor its complement contains a chordless cycle with a number of vertices greater than 4.

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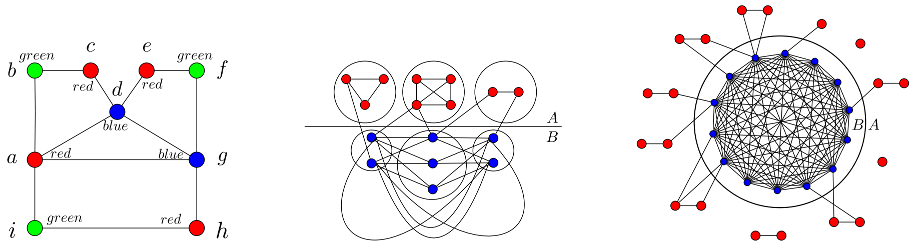


(a) A weakly chordal graph with an optimal 3-clique-colouring

(b) A $(4,3)$-polar graph

(c) A generalized split graph, which is a $(\alpha,1)$-polar graph, for fixed $\alpha \geq 2$

Fig. 1. Examples of (α, β)-polar graphs.

Both clique-colouring and perfect graphs have attracted much attention due to a conjecture posed by Duffus et al. [5] that *perfect graphs are k-clique-colourable for some constant k*. This conjecture has not yet been proved. Following the chronological order, Kratochvíl and Tuza gave a framework to argue that 2-clique-colouring is $\mathcal{NP}$-hard and proved that 2-clique-colouring is $\mathcal{NP}$-complete for K_4-free perfect graphs [7]. Notice that K_3-free perfect graphs are bipartite graphs, which are clearly 2-clique-colourable. Moreover, 2-clique-colouring is in Σ_2^P, since it is co$\mathcal{NP}$ to check that a colouring of the vertices is a clique-colouring. A few years later, the 2-clique-colouring problem was proved to be a Σ_2^P-complete problem by Marx [9], a major breakthrough in the clique-colouring area. Défossez [4] proved later that 2-clique-colouring of perfect graphs remained a Σ_2^P-complete problem.

When restricted to chordal graphs, 2-clique-colouring is in $\mathcal{P}$, since all chordal graphs are 2-clique-colourable [10]. Notice that chordal graphs are a subclass of weakly chordal graphs, while perfect graphs are a superclass of weakly chordal graphs. In contrast to chordal graphs, not all weakly chordal graphs are 2-clique-colourable (see Fig. 1a).

We show that 2-clique-colouring of weakly chordal graphs is a Σ_2^P-complete problem, improving the proof of Défossez [4] that 2-clique-colouring is a Σ_2^P-complete problem for perfect graphs. As a remark, Défossez [4] constructed a graph which is not a weakly chordal graph as long as it has chordless cycles with even number of vertices greater than 5 as induced subgraphs. We determine a hierarchy of nested subclasses of weakly chordal graphs whereby each graph class is in a distinct complexity class, namely Σ_2^P-complete, $\mathcal{NP}$-complete, and $\mathcal{P}$.

A graph is *(α, β)-polar* if there exists a partition of its vertex set into two sets A and B such that all connected components of the subgraph induced by A and of the complementary subgraph induced by B are complete graphs. Moreover, the order of each connected component of the subgraph induced by A (resp. of the complementary subgraph induced by B) is upper bounded by α (resp. upper bounded by β) [2]. A *satellite* of an (α, β)-polar graph is a connected component of the subgraph induced by A (see Fig. 1b). In this work, we restrict ourselves to the *(α, β)-polar* graphs with $\beta = 1$, so the subgraph induced by B is complete and the order of each satellite is upper bounded by α (see Fig. 1c). Clearly, $(\alpha, 1)$-polar graphs are perfect, since they do not contain chordless cycles with an odd number of vertices greater than 4 nor their complements [11].

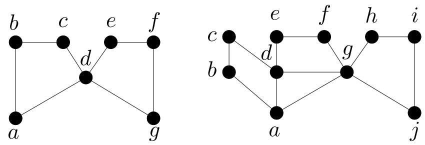
A *generalized split graph* is a graph G such that G or its complement is an $(\alpha, 1)$-polar graph, for some α [11]. See Fig. 1c for an example of a generalized split graph, which is a (2, 1)-polar graph. The class of generalized split graphs plays an important role in the areas of perfect graphs and clique-colouring. This class was introduced by Prömel and Steger [11] to show that the strong perfect graph conjecture is at least asymptotically true by proving that almost all C_5-free graphs are generalized split graphs. Approximately 14 years later the strong perfect graph conjecture became the *Strong Perfect Graph Theorem* by Chudnovsky et al. [3]. Regarding clique-colouring, Bacsó et al. [1] proved that generalized split graphs are 3-clique-colourable and concluded that almost all perfect graphs are 3-clique-colourable [1]. This conclusion supports the conjecture due to Duffus et al. [5]. In fact, there is no example of a perfect graph where more than three colours would be necessary to clique-colour. Surprisingly, after more than 20 years, relatively little progress has been made on the conjecture.

The class of $(\alpha, 1)$-polar graphs, for fixed $\alpha \geq 3$, is incomparable to the class of weakly chordal graphs. Indeed, the chordless path with seven vertices P_7 is weakly chordal but not generalized split, and the complement of the chordless cycle with six vertices $\overline{C_6}$ is (3,1)-polar but by definition is not weakly chordal. Clearly, $(\alpha, 1)$-polar graphs do not contain all chordless cycles with at least 5 vertices. For complements of chordless cycles with an even number of $2t$ vertices, the graph is $(t, 1)$-polar but not $(t-1, 1)$-polar. Hence, $(2, 1)$-polar graphs are a subclass of weakly chordal graphs. We show that 2-clique-colouring of $(2, 1)$-polar graphs is a $\mathcal{NP}$-complete problem. Finally, the class of $(1, 1)$-polar graphs is precisely the class of split graphs. It is interesting to recall that 2-clique-colouring of $(1, 1)$-polar graphs is in $\mathcal{P}$, since $(1, 1)$-polar are a subclass of chordal graphs, which are 2-clique-colourable.

Giving continuity to our results, we investigate an open problem left by Kratochvíl and Tuza [7] to determine the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3. Restricting the size of the cliques to be at least 3, we first show that 2-clique-colouring remains $\mathcal{NP}$-complete for $(3, 1)$-polar graphs. Recall that $(3, 1)$-polar graphs with all cliques having size at least 3 are weakly chordal, since $\overline{C_6}$ has cliques of size 2 and $\overline{C_{2t}}$ for $t > 3$ is not

Table 1
2-clique-colouring complexity of perfect graphs and subclasses.

Class			2-clique-colouring complexity
–	Perfect	–	Σ_2^P-complete [4]
		K_4-free	$\mathcal{NP}$-complete [7]
		K_3-free (Bipartite)	$\mathcal{P}$
	Weakly chordal	–	Σ_2^P-complete
	(3, 1)-polar	–	$\mathcal{NP}$-complete
	(2, 1)-polar	–	
	Chordal (includes Split)	–	$\mathcal{P}$ [10]
All cliques having size at least 3	Perfect	–	Σ_2^P-complete
	Weakly chordal	–	
		(3, 1)-polar	$\mathcal{NP}$-complete
	(2, 1)-polar	–	$\mathcal{P}$



(a) $AK(a,g)$ (b) $NAS(a,j)$

Fig. 2. Auxiliary graphs $AK(a, g)$ and $NAS(a, j)$.

(3, 1)-polar. Subsequently, we prove that the 2-clique-colouring of (2, 1)-polar graphs becomes polynomial when all cliques have size at least 3. Recall that the 2-clique-colouring of (2, 1)-polar graphs is $\mathcal{NP}$-complete when there are no restrictions on the size of the cliques.

We finish the paper answering the open problem of determining the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3 [7], by improving our proof that 2-clique-colouring is a Σ_2^P-complete problem for weakly chordal graphs. We replace each K_2 clique by a gadget with no clique of size 2, which forces distinct colours into two given vertices.

The paper is organized as follows. In Section 2, we show that 2-clique-colouring is still Σ_2^P-complete for weakly chordal graphs. We then determine a hierarchy of nested subclasses of weakly chordal graphs whereby each graph class is in a distinct complexity class, namely Σ_2^P-complete, $\mathcal{NP}$-complete, and $\mathcal{P}$. In Section 3, we determine the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3, answering a question of Kratochvíl and Tuza [7]. We then determine a hierarchy of nested subclasses of perfect graphs with all cliques having size at least 3 whereby each graph class is in a distinct complexity class. We refer the reader to Table 1 for our results and related work about 2-clique-colouring complexity of perfect graphs, and to Fig. 9 for a clear overview of the different considered classes.

2. Hierarchical complexity of 2-clique-colouring of weakly chordal graphs

Défossez proved that 2-clique-colouring of perfect graphs is a Σ_2^P-complete problem [4]. In this section, we strengthen this result by showing that the problem remains Σ_2^P-complete for weakly chordal graphs. We show a subclass of perfect graphs (resp. of weakly chordal graphs) in which 2-clique-colouring is neither a Σ_2^P-complete problem nor in $\mathcal{P}$ – assuming that the polynomial hierarchy does not collapse – namely (3, 1)-polar graphs (resp. (2, 1)-polar graphs). Recall that 2-clique-colouring of (1, 1)-polar graphs is in $\mathcal{P}$, since (1, 1)-polar are a subclass of chordal graphs, thereby 2-clique-colourable. Notice that weakly chordal, (2, 1)-polar, and (1, 1)-polar (resp. perfect, (3, 1)-polar, and (1, 1)-polar) are nested classes of graphs.

Given a graph $G = (V, E)$ and vertices $a, g \in V$, we say that we add to G a copy of an auxiliary graph $AK(a, g)$ of order 7 – depicted in Fig. 2a – if we change the definition of G by doing the following: we first change the definition of V by adding to it copies of the five vertices b, c, d, e, and f of the auxiliary graph $AK(a, g)$; then we change the definition of E by adding to it copies of the eight edges (a, b), (b, c), (c, d), (d, e), (e, f), (f, g), (a, d), and (d, g) of $AK(a, g)$. Similarly, given a graph $G = (V, E)$ and vertices $a, j \in V$, we say that we add to G a copy of an auxiliary graph $NAS(a, j)$ of order 10 – depicted in Fig. 2b – if we change the definition of G by doing the following: we first change the definition of V by adding to it copies of the eight vertices b, c, d, e, f, g, h, and i of the auxiliary graph $NAS(a, j)$; then we change the definition of E by adding to it copies of the thirteen edges (a, b), (b, c), (c, d), (d, e), (e, f), (f, g), (a, d), (a, g), (d, g), (g, j), (g, h), (h, i), and (i, j) of $NAS(a, j)$.

The auxiliary graph $AK(a, g)$ is constructed to force the same colour (in a 2-clique-colouring) to vertices a and g, while the auxiliary graph $NAS(a, j)$ is constructed to force distinct colours (in a 2-clique-colouring) to vertices a and j (see Lemmas 1 and 2).

Lemma 1. *Let G be a graph and a, g be vertices in G. If we add to G a copy of an auxiliary graph $AK(a, g)$, then in any 2-clique-colouring of the resulting graph, vertices a and g have the same colour.*

Proof. Follows from the fact that in $AK(a, g)$ there exists a path $abcdefg$ such that no edge lies in a triangle of G. □

Lemma 2. *Let G be a graph and a, j be vertices in G. If we add to G a copy of an auxiliary graph $NAS(a, j)$, then in any 2-clique-colouring of the resulting graph, vertices a and j have distinct colours.*

Proof. Follows from the fact that in $NAS(a, j)$ there exists a path $abcdefghij$ such that no edge lies in a triangle of G. □

We improve the proof of Défossez [4], in order to determine the complexity of 2-clique-colouring for weakly chordal graphs. Consider the QSAT2 problem, which is the Σ_2^P-complete canonical problem [9], as follows.

Problem 1. Quantified 2-Satisfiability (QSAT2)
Instance: *A formula $\Psi = (X, Y, D)$ composed of a disjunction D of implicants (that are conjunctions of three literals) over two sets X and Y of variables.*
Question: *Is there a truth assignment for X such that for every truth assignment for Y the formula is true?*

We prove that 2-clique-colouring weakly chordal graphs is Σ_2^P-complete by reducing the Σ_2^P-complete canonical problem QSAT2 to it. For a QSAT2 formula $\Psi = (X, Y, D)$, a weakly chordal graph G is constructed such that graph G is 2-clique-colourable if, and only if, there is a truth assignment of X, such that Ψ is true for every truth assignment of Y.

Theorem 3. *The problem of 2-clique-colouring is Σ_2^P-complete for weakly chordal graphs.*

Proof. The problem of 2-clique-colouring is in Σ_2^P, since it is co$\mathcal{NP}$ to check that a colouring of the vertices is a 2-clique-colouring.

We prove that 2-clique-colouring weakly chordal graphs is Σ_2^P-hard by reducing QSAT2 to it. Let n, m, and p be the number of variables X, Y, and implicants, respectively, in formula Ψ. We define graph G, as follows.

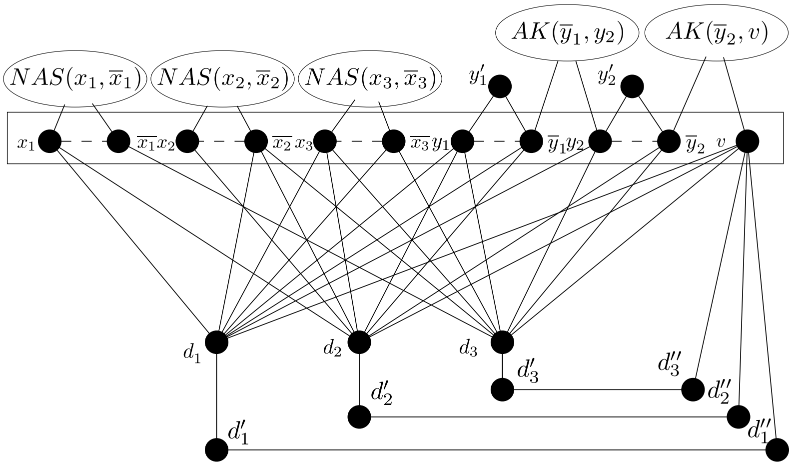
- for each variable x_i, we create vertices x_i and $\overline{x}_i$;
- for each variable y_j, we create vertices y_j, y'_j, and $\overline{y}_j$ and edges $y_j y'_j$, and $y'_j \overline{y}_j$;
- we create a vertex v and edges so that the set $\{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n, y_1, \overline{y}_1, \ldots, y_m, \overline{y}_m, v\}$ induces a complete subgraph of G minus the matching $\{\{x_1, \overline{x}_1\}, \ldots, \{x_n, \overline{x}_n\}, \{y_1, \overline{y}_1\}, \ldots, \{y_m, \overline{y}_m\}\}$;
- add copies of the auxiliary graph $NAS(x_i, \overline{x}_i)$, for $i = 1, \ldots, n$;
- add copies of the auxiliary graph $AK(\overline{y}_j, y_{j+1})$, for $j = 1, \ldots, m-1$;
- add a copy of $AK(\overline{y}_m, v)$; and
- for each implicant d_k, we create vertices d_k, d'_k, d''_k, and we add the edges $d_k d'_k$, $d'_k d''_k$, $d''_k v$, and $d_k v$. Moreover, each vertex d_k is adjacent to vertex v and to a vertex l in $\{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n, y_1, \overline{y}_1, \ldots, y_m, \overline{y}_m\}$ if, and only if, the literal corresponding to $\overline{l}$ is not in the implicant corresponding to vertex d_k.

Refer to Fig. 3 for an example of such construction, given a formula $\Psi = (x_1 \wedge \overline{x}_2 \wedge y_2) \vee (x_1 \wedge x_3 \wedge \overline{y}_2) \vee (\overline{x}_1 \wedge \overline{x}_2 \wedge y_1)$.

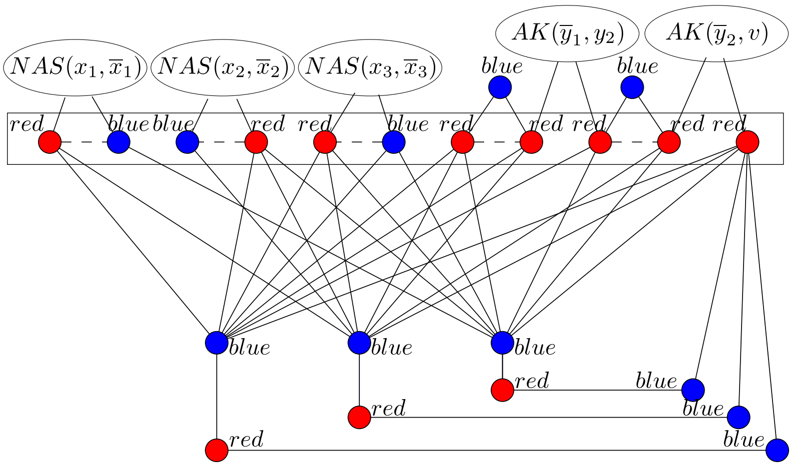
We claim that graph G is 2-clique-colourable if, and only if, Ψ has a solution. Assume there exists a valuation v_X of X such that Ψ is satisfied for any valuation of Y. We give a colouring to the graph G, as follows.

- assign colour 1 to y_j, $\overline{y}_j$, d'_k, and v,
- assign colour 2 to y'_j, d_k and d''_k,
- extend the unique 2-clique-colouring to the $m-1$ copies of the auxiliary graph $AK(\overline{y}_j, y_{j+1})$ and $AK(\overline{y}_m, v)$,
- assign colour 1 to x_i if the corresponding variable is *true* in v_X, otherwise we assign colour 2 to it,
- assign colour 2 to $\overline{x}_i$ if the corresponding variable is *true* in v_X, otherwise we assign colour 1 to it,
- extend the unique 2-clique-colouring to the n copies of the auxiliary graph $NAS(x_i, \overline{x}_i)$.

We prove that this is indeed a 2-clique-colouring. Let us assume that it is not the case and that there exists a clique K of G that is monochromatic. Clearly, K is not contained in a copy of any auxiliary graph, and it does not contain any vertex



(a) Graph constructed for a QSAT2 instance $\Psi = (x_1 \wedge \overline{x}_2 \wedge y_2) \vee (x_1 \wedge x_3 \wedge \overline{y}_2) \vee (\overline{x}_1 \wedge \overline{x}_2 \wedge y_1)$



(b) A satisfying truth assignment of $x_1 = \overline{x_2} = x_3 = T$

Fig. 3. Example of a graph constructed for a QSAT2 instance, where *NAS* and *AK* denote the respectively auxiliary graphs.

of type y'_j, d'_k, or d''_k. As v is adjacent to all other vertices (which are the x_i, $\overline{x}_i$, y_j, $\overline{y}_j$, and d_k), we deduce that $v \in K$ and, subsequently, that all vertices of K have colour 1. Moreover, K contains exactly one vertex among x_i and $\overline{x}_i$, i.e. the one corresponding to the literal which is *true* in ν_X, and similarly exactly one vertex among y_j and $\overline{y}_j$. We remark that K does not contain any d_k since they have colour 2. Then we define a valuation ν_Y in the following way. If $y_j \in K$, then ν_Y assigns value *true* to the corresponding variable, otherwise ν_Y assigns the value *false*. Thus, the literals corresponding to the vertices of $K \setminus \{v\}$ are exactly those that are *true* in the total valuation (ν_X, ν_Y). Let us consider now any d_k. Since K is a clique, each d_k is not adjacent to at least one vertex of K. By construction of G, this means that all implicants are *false*, which contradicts the definition of ν_X. Hence, there is no monochromatic clique and we have a 2-clique-colouring.

For the converse, we now assume that G is 2-clique-colourable. For every i, the vertices x_i and $\overline{x}_i$ have opposite colours in any 2-clique-colouring of G (see Lemma 2). The set $\{y_1, y_2, \ldots, y_m, \overline{y}_1, \overline{y}_2, \ldots, \overline{y}_m, v\}$ is monochromatic. Indeed, the three vertices y_j, $\overline{y}_j$, and y_{j+1} have the same colour, since sets $\{y_j, y'_j\}$ and $\{y'_j, \overline{y}_j\}$ are cliques and, by Lemma 1, vertices $\overline{y}_j$ and y_{j+1}, as well as vertices $\overline{y}_m$ and v, have the same colour. Finally, since each path $d_k d'_k d''_k v$ is chordless, vertices $d_1, d_2, \ldots, d_p$ all have the same colour, which is the opposite to the colour of v. Consider a 2-clique colouring of G using colours 1 and 2 and assume, w.l.o.g., that v has colour 1. Then, y_j and $\overline{y}_j$ have colour 1, and d_k has colour 2. Vertices x_i and $\overline{x}_i$ have opposite colours and we define ν_X in the following way. The literal x_i is assigned *true* in ν_X if the corresponding vertex has colour 1 in the clique-colouring, otherwise it is assigned *false* in ν_X. Let ν_Y be any valuation of Y. Consider the set of vertices K that contains v and the vertices corresponding to literals which are *true* in the total valuation (ν_X, ν_Y). Clearly, set K induces a complete subgraph. Since all those vertices have colour 1 and we have a 2-clique-colouring, it follows that K is not a clique. As a consequence, there exists some d_k which is adjacent to all vertices of K. Thus, the

Algorithm 1: Polynomial-time algorithm to check if no clique intersecting a given satellite A_i of an $(\alpha, 1)$-polar graph G, for a fixed $\alpha \geq 1$, is monochromatic.

input : an $(\alpha, 1)$-polar graph G with partition (A, B)
a 2-colouring π of G
a satellite $A_i = \{x_1, x_2, \ldots, x_{|A_i|}\}$ of G
output: *YES*, if every clique of G that intersects A_i is polychromatic; *NO*, otherwise

foreach *non-empty subset* $X_j \subset A_i$ **do**
  $S_j := X_j \cup B$
  foreach *element* $p \in X_j$ **do**
    $S_j := S_j \cap N[p]$
  maximal := *YES*
  foreach *element* $q \in A_i \setminus X_j$ **do**
    if $N_B(q) == S_j \cap B$ **then**
      maximal := *NO*
  if *maximal* == *yes and* $|\pi(S_j)| = 1$ **then**
    return *NO*
Return *YES*

corresponding implicant is *true* in that valuation and this proves that Ψ is satisfied for any valuation v_Y and that v_X has the right property.

It now remains to be proved that G is a weakly chordal graph. Suppose that G contains W a chordless cycle of size at least 5. Suppose that $d'_k \in W$, which implies $d_k, d'_k, d''_k \in W$, and that $v \in W$, a contradiction to the size of W. An analogous argument holds for the hypothesis $d''_k \in W$. Hence, W does not contain d'_k, d''_k. Recall that by construction of G, the set $S = \{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n, y_1, \overline{y}_1, \ldots, y_m, \overline{y}_m, v\}$ induces a complete subgraph minus the matching $\{\{x_1, \overline{x}_1\}, \ldots, \{x_n, \overline{x}_n\}, \{y_1, \overline{y}_1\}, \ldots, \{y_m, \overline{y}_m\}\}$. Moreover, $G \setminus (S \cup \{d'_1, \ldots, d'_p, d''_1, \ldots, d''_p\})$ is disconnected, and has as connected components singletons d_k, or singletons y'_j, or chordless paths P_5, or connected components whose largest cycle is a C_4. Therefore, W must contain at least two non-adjacent vertices of set S, a contradiction to the size of W, since any pair of non-adjacent vertices in S are extremities of chordless paths of size at most 2. Now suppose that G contains W the complement of a chordless cycle of size at least 6, where $W = w_1 w_2 w_3 \ldots w_t$, with $t \geq 6$ and $w_i w_{i+1} \notin E$. Every vertex of W has degree larger than 2 in G, so W does not contain d'_k, d''_k, y'_j, nor vertices of $AK(a, g) \setminus \{a, g\}$, and nor vertices of $NAS(a, j) \setminus \{a, j\}$. Moreover, W does not contain v, a vertex adjacent to every vertex of set $S \setminus v$. Since $R = \{d_1, d_2, \ldots, d_p\}$ is an independent set, $|W \cap R| \leq 2$, and in case $|W \cap R| = 2$, the two vertices are consecutive in W. On the other hand, recall $S \setminus v$ is an antimatching, and $|W \cap (S \setminus v)| \geq 4$ forces S to have at least one vertex s with two vertices in S that are non-adjacent to vertex s, and a contradiction. □

Now, our focus is on showing a subclass of weakly chordal graphs, actually, a subclass of generalized split graphs, in which 2-clique-colouring is $\mathcal{NP}$-complete, namely (2, 1)-polar graphs.

Défossez [4] showed that it is co$\mathcal{NP}$-complete to check whether a 2-colouring of a complement of a bipartite graph is a 2-clique-colouring [4]. Clearly, complements of bipartite graphs are a subclass of generalized split graphs. We show next that if α is fixed, listing all cliques of a $(\alpha, 1)$-polar graph and checking if each clique is polychromatic can be done in polynomial-time – although the constant behind the big O notation is impracticable – which implies that the problem of 2-clique-colouring is in $\mathcal{NP}$ for $(\alpha, 1)$-polar graphs, where α is fixed.

The outline of the algorithm follows. We create a subroutine in which, given a satellite K of G, we check whether every clique of G containing a subset of K is polychromatic. Lemma 4 determines the complexity of the subroutine and proves its correctness. The algorithm runs the subroutine for each satellite of G and, as a final step, check whether part B is polychromatic if, and only if, part B is a clique of G. Theorem 5 determines the complexity of the algorithm and prove its correctness.

Lemma 4. *Let G be an $(\alpha, 1)$-polar graph, let π be a 2-colouring of G, and let A_i be a satellite of G. Algorithm 1 returns NO if there exists a monochromatic clique of G that intersects A_i, and YES otherwise. Moreover, Algorithm 1 runs in polynomial time.*

Proof. We prove the correctness of Algorithm 1. The algorithm has an external loop, that is executed for each subset X_j of A_i, verifies whether G has a clique intersecting A_i precisely in X_j, and if this clique is monochromatic. It is easy to see that Algorithm 1 runs in polynomial time. The external loop of Algorithm 1 runs at most 2^{α} times, while each of the internal loops is executed a number of times that is bounded by n. □

Theorem 5. *Let G be an $(\alpha, 1)$-polar graph and let π be a 2-colouring of G. Algorithm 2 returns YES if π is a 2-clique-colouring, and NO otherwise. Moreover, Algorithm 2 runs in polynomial time.*

Algorithm 2: Polynomial-time algorithm to check if a 2-colouring of an $(\alpha, 1)$-polar graph, for a fixed $\alpha \geq 1$, is a valid clique-colouring.

```
input  :  an (α, 1)-polar graph G with partition (A, B)
          a 2-colouring π of G
output: YES, if every clique of G is polychromatic; NO, otherwise

begin
    foreach satellite A_i of A do
        answer == Algorithm 1(G, π, A_i) if answer == NO then
            return NO
    if |π(B)| == 2 then
        return YES
    else
        foreach v ∈ A do
            if N_B(v) == B then
                return YES
        return NO
```

Proof. The correctness of Algorithm 2 follows. A clique of G contains at least one vertex of a satellite of G or it is B. The first loop of Algorithm 2 checks whether there exists some monochromatic clique intersecting any of the satellites of G – if one such monochromatic clique is found, then the algorithm stops and returns *NO*. If the algorithm passes this first loop, then the only possible monochromatic clique is B. Two tests are done. First, the algorithm checks whether B uses two colours: if two colours appear in B then there is no monochromatic clique and the algorithm returns *YES*. If B is monochromatic, then it remains to check if B is a clique. This is done in the last loop, which tests if some vertex in A is adjacent to every vertex in B; if such a vertex if found then B is not maximal and Algorithm 2 returns *YES*, otherwise B is a monochromatic clique and Algorithm 2 returns *NO*.

It is easy to see that Algorithm 2 runs in polynomial time. The first loop of Algorithm 2 runs at most n times the Algorithm 1, which in turn runs in polynomial time. The second loop is executed at most n times. □

Once we have proved that 2-clique-colouring is in $\mathcal{NP}$ for $(\alpha, 1)$-polar graphs, for fixed α, we proceed to show that 2-clique-colouring is $\mathcal{NP}$-hard for $(2, 1)$-polar graphs.

Theorem 6. *The problem of 2-clique-colouring is $\mathcal{NP}$-complete for (2, 1)-polar graphs.*

Proof. The problem of 2-clique-colouring a (2, 1)-polar graph is in $\mathcal{NP}$: Theorem 5 confirms that it is in $\mathcal{P}$ to check whether a 2-colouring of a (2, 1)-polar graph is a 2-clique-colouring.

We prove that 2-clique-colouring (2, 1)-polar graphs is $\mathcal{NP}$-hard by reducing hypergraph 2-colouring [8] to it. For every hypergraph $\mathcal{H}$, a (2, 1)-polar graph G is constructed such that hypergraph $\mathcal{H}$ is 2-colourable if, and only if, graph G is 2-clique-colourable. Let n (resp. m) be the number of vertices (resp. hyperedges) in hypergraph $\mathcal{H}$. We define graph $G = (V, E)$, as follows.

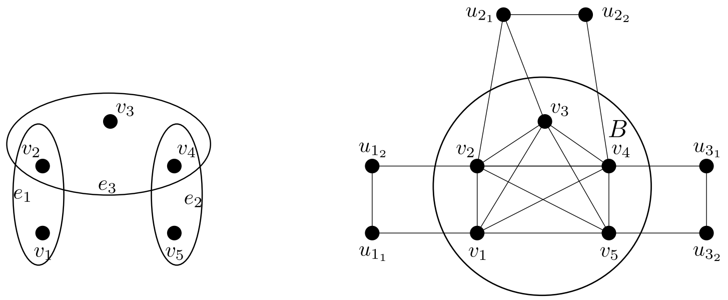
- for each vertex v_i, $1 \leq i \leq n$, of hypergraph $\mathcal{H}$, we create a vertex v_i in the vertex set V of graph G, so that the set $\{v_1, \dots, v_n\}$ induces a complete subgraph of G, which is the part B of the partition (A, B) of the vertex set V;
- for each hyperedge $e_j = \{v_1, \dots, v_l\}$, $l > 1$, $1 \leq j \leq m$, we create two vertices u_{j_1} and u_{j_2} in V. Moreover, we create edges $u_{j_1}u_{j_2}$, $u_{j_1}v_1, \dots, u_{j_1}v_{l-1}$, and $u_{j_2}v_l$ so that $S_j = \{u_{j_1}, u_{j_2}\}$ is a satellite satisfying $N_B(u_{j_1}) \cap N_B(u_{j_2}) = \emptyset$.

Clearly, G is a (2, 1)-polar graph and such construction is done in polynomial-time. Refer to Fig. 4 for an example of such construction. Note that every satellite S of the constructed (2, 1)-polar graph G satisfies $S = \{s_1, s_2\}$ and $N_B(s_1) \cap N_B(s_2) = \emptyset$.

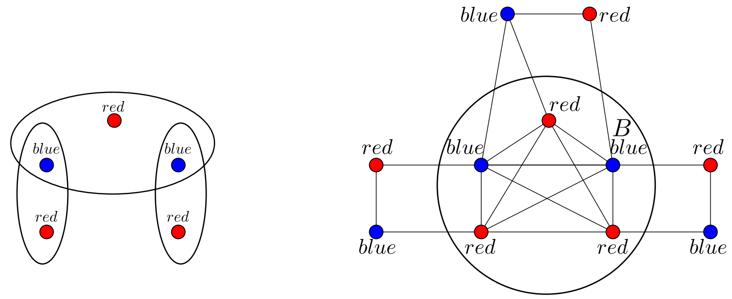
We claim that hypergraph $\mathcal{H}$ is 2-colourable if, and only if, graph G is 2-clique-colourable.

Assume first that there exists a proper 2-colouring π of $\mathcal{H}$. We give a 2-clique-colouring to the constructed graph G, as follows.

- assign colour $\pi(v)$ for each corresponding vertex v of part B of graph G. Note that for each satellite S_j, $1 \leq j \leq m$, we have that $S_j = \{u_{j_1}, u_{j_2}\}$ satisfies $N_B(u_{j_1}) \cup N_B(u_{j_2})$ is polychromatic, since this set of vertices of V corresponds to a hyperedge in $\mathcal{H}$.
- assign colour $\pi(u)$ for each vertex u of part A of graph G as follows. Let $S_j = \{u_{j_1}, u_{j_2}\}$ be a satellite. Assume $N_B(u_{j_1})$ is polychromatic. Then, assign colour 1 to u_{j_2} if there is a vertex in $N_B(u_{j_2})$ coloured 2, and assign colour 2 to u_{j_2} if there is a vertex in $N_B(u_{j_2})$ coloured 1; next assign the other colour to u_{j_1}. Otherwise, assume both $N_B(u_{j_1})$ and $N_B(u_{j_2})$ are monochromatic, coloured 1 and 2, respectively, and assign u_{j_1} colour 2 and u_{j_2} colour 1.



(a) Hypergraph instance and its corresponding constructed (2, 1)-polar graph



(b) A proper 2-colouring assignment of the hypergraph and its corresponding 2-clique-colouring assignment of the constructed graph, as well as a 2-clique-colouring assignment of the constructed graph and its corresponding proper 2-colouring assignment of the hypergraph

Fig. 4. Example of a (2, 1)-polar graph constructed for a given hypergraph instance.

The above colouring is indeed a 2-clique-colouring. Consider the part $B = \{v_1, v_2, \ldots, v_n\}$ of the partition (A, B) of the vertex set V. Clearly, the above colouring assigns 2 colours to this set. Moreover, for each satellite S_j, $1 \leq j \leq m$, both cliques $u_{j_1} \cup N_B(u_{j_1})$ and $u_{j_2} \cup N_B(u_{j_2})$ are 2-coloured, and the clique $\{u_{j_1}, u_{j_2}\}$ is 2-coloured.

For the converse, we now assume that G is 2-clique-colourable and we consider any 2-clique-colouring π' of G. We give a proper 2-colouring to hypergraph $\mathcal{H}$, as follows. Assign colour $\pi'(v)$ for each vertex v of $\mathcal{H}$. It is enough to prove that each satellite $S_j = \{u_{j_1}, u_{j_2}\}$ satisfies that $N_B(u_{j_1}) \cup N_B(u_{j_2})$ is polychromatic, since each hyperedge in $\mathcal{H}$ corresponds to the neighbourhood in B of a satellite. Suppose that every vertex of $N_B(u_{j_1}) \cup N_B(u_{j_2})$ has colour 1. Since both $u_{j_1} \cup N_B(u_{j_1})$ and $u_{j_2} \cup N_B(u_{j_2})$ are cliques, we have that both u_{j_1} and u_{j_2} have colour 2, and now the satellite $S_j = \{u_{j_1}, u_{j_2}\}$ is a monochromatic clique, a contradiction. □

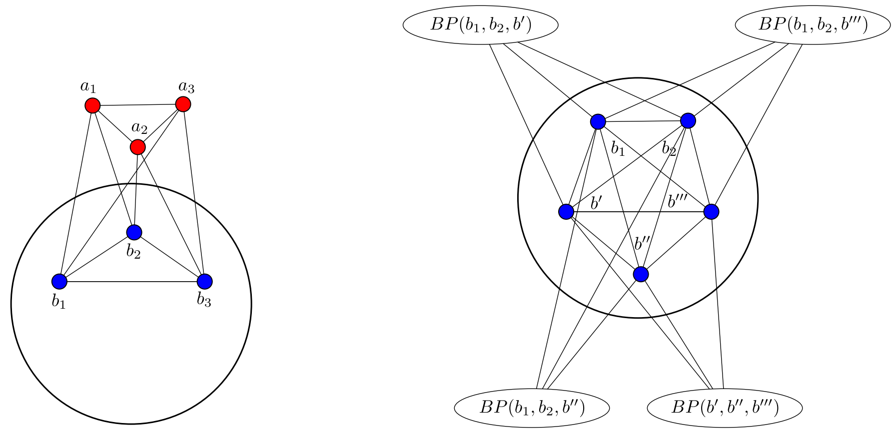
3. Restricting the size of the cliques

Kratochvíl and Tuza [7] are interested in determining the complexity of 2-clique-colouring of perfect graphs with all cliques having size at least 3. We determine what happens with the complexity of 2-clique-colouring of (2, 1)-polar graphs, of (3, 1)-polar graphs, and of weakly chordal graphs, respectively, when all cliques are restricted to have size at least 3. The latter result address Kratochvíl and Tuza's question.

Given graph $G = (V, E)$ and mutually adjacent vertices $b_1, b_2, b_3 \in V$, we say that we add to G a copy of an auxiliary graph $BP(b_1, b_2, b_3)$ of order 6 – depicted in Fig. 5a – if we change the definition of G by doing the following: we first change the definition of V by adding to it copies of the vertices a_1, a_2, a_3 of the auxiliary graph $BP(b_1, b_2, b_3)$; second, we change the definition of E by adding to it copies of the 9 edges (a_1, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_3), (a_1, a_2), (a_1, a_3), (a_2, a_3) of $BP(b_1, b_2, b_3)$.

Similarly, given a graph $G = (V, E)$, a set of mutually adjacent vertices D, and adjacent vertices $b_1, b_2 \in D$, we say that we add to (G, D) a copy of an auxiliary graph $BS(b_1, b_2)$ of order 17 – depicted in Fig. 5b – if we change the definition of G by doing the following: we first change the definition of V by adding to it copies of the vertices b', b'', b''' of the auxiliary graph $BS(b_1, b_2)$; second, we change the definition of E by adding to it edges so that b', b'', b''' and the vertices in D are mutually adjacent; and finally, we add copies of the auxiliary graphs $BP(b_1, b_2, b')$, $BP(b_1, b_2, b'')$, $BP(b_1, b_2, b''')$, and $BP(b', b'', b''')$.

Lemma 7. *Let $G = (V, E)$ be a weakly chordal graph (resp. (3, 1)-polar graph, with vertex partition (A, B)) and let mutually adjacent vertices $b_1, b_2, b_3 \in V$ (resp. $b_1, b_2, b_3 \in B$). If we add to G a copy of an auxiliary graph $BP(b_1, b_2, b_3)$, then the following assertions about the resulting graph G' are true.*



(a) Auxiliary graph $BP(b_1, b_2, b_3)$ (b) Auxiliary graph $BS(b_1, b_2)$

Fig. 5. Auxiliary graphs $BP(b_1, b_2, b_3)$ and $BS(b_1, b_2)$.

- *The resulting graph G' is weakly chordal (resp. (3, 1)-polar).*
- *If all cliques of G have size at least 3, then all cliques of G' have size at least 3.*
- *Any 2-clique-colouring of G' uses 2 colours in the set of vertices $\{b_1, b_2, b_3\}$.*
- *If G' is 2-clique-colourable, then G is 2-clique-colourable.*
- *If there exists a 2-clique-colouring of G that uses 2 colours in the set of vertices $\{b_1, b_2, b_3\}$, then G' is 2-clique-colourable.*

Proof. Let $G = (V, E)$ be a weakly chordal graph and let mutually adjacent vertices $b_1, b_2, b_3 \in V$. Add to G a copy of an auxiliary graph $BP(b_1, b_2, b_3)$ in order to obtain graph G'.

Suppose, by contradiction, that G' is not weakly chordal, which means that G' has W a chordless cycle with a number of vertices greater than 4 or the complement of a chordless cycle with a number of vertices greater than 5. Clearly, the auxiliary graph $BP(b_1, b_2, b_3)$ is a weakly chordal graph. Since both G and $BP(b_1, b_2, b_3)$ are weakly chordal graphs, a chordless cycle W with a number of vertices greater than 4 contains a vertex of $\{a_1, a_2, a_3\}$ and a vertex of $G' \setminus BP(b_1, b_2, b_3)$. Note that $\{b_1, b_2, b_3\}$ induces a complete graph and is a cutset of G' that disconnects $\{a_1, a_2, a_3\}$ from $G' \setminus BP(b_1, b_2, b_3)$. So, every cycle with vertices of $\{a_1, a_2, a_3\}$ and of $G' \setminus BP(b_1, b_2, b_3)$ contains a chord, i.e. there is no such chordless cycle W. For W the complement of a chordless cycle with a number of vertices greater than 5, note that a_1, a_2, and a_3 have each exactly 4 neighbours, so $|W| \in \{6, 7\}$. If W has only one vertex a_i of $\{a_1, a_2, a_3\}$, then a_i has at most 2 neighbours in W, which is a contradiction. If W has only two vertices a_i, a_j, $i \neq j$, of $\{a_1, a_2, a_3\}$, then W contains $\{b_1, b_2, b_3\}$, otherwise a_i or a_j have at most 2 neighbours in W. Let u be a vertex of $G' \setminus BP(b_1, b_2, b_3)$ in W. Since neither a_i nor a_j are neighbours of u and a vertex in W has at most two non-neighbours in W, set $\{u, b_1, b_2, b_3\}$ induces a complete subgraph, which is a contradiction. If W has all three vertices a_1, a_2, a_3, then a vertex of $G' \setminus BP(b_1, b_2, b_3)$ in W has 3 non-neighbours in W, which is a contradiction. Hence, there is no such W and G' is weakly chordal. If G is a (3, 1)-polar graph with partition (A, B), and $b_1, b_2, b_3 \in B$, then G' is a (3, 1)-polar graph with partition (A', B'), where $A' = A \cup \{a_1, a_2, a_3\}$ and $B' = B$ as the partition of $V(G')$ into two sets. Notice that the added satellite is a triangle. Hence, G' is a (3, 1)-polar graph.

The cliques of G' that are not cliques of G are $\{a_1, b_1, b_2\}$, $\{a_2, b_2, b_3\}$, $\{a_3, b_1, b_3\}$, $\{a_1, a_2, b_2\}$, $\{a_2, a_3, b_3\}$, $\{a_1, a_3, b_1\}$, and $\{a_1, a_2, a_3\}$. Clearly, if all cliques of G have size at least 3, then all cliques of G' have size at least 3.

Since $\{a_1, a_2, a_3\}$ is a clique of G', any 2-clique-colouring π' of G' uses 2 colours in the set $\{a_1, a_2, a_3\}$. Let $i, j, k, \ell \in \{1, 2, 3\}$ and $\pi'(a_i) \neq \pi'(a_j)$. Since $\{a_i, b_i, b_k\}$ (resp. $\{a_j, b_j, b_\ell\}$) is a clique of G', π' assigns a colour which is not $\pi'(a_i)$ to b_i or b_k (resp. π' assigns a colour which is not $\pi'(a_j)$ to b_j or b_ℓ). Hence, π' uses 2 colours in the set $\{b_1, b_2, b_3\}$.

Note that π' is also a 2-clique-colouring of G, since $\mathcal{C}(G) \subset \mathcal{C}(G')$. Now, consider π a 2-clique-colouring of G that uses 2 colours in the set $\{b_1, b_2, b_3\}$. It is easy to extend π in order to assign colours to the vertices of $BP(b_1, b_2, b_3) \setminus G$, such that all cliques of $\mathcal{C}(G') \setminus \mathcal{C}(G)$ are polychromatic. □

Lemma 8. *Let $G = (V, E)$ be a weakly chordal graph (resp. (3, 1)-polar graph with vertex partition (A, B)) and let adjacent vertices $b_1, b_2 \in V$ (resp. $b_1, b_2 \in B$). If we add to $(G, \{b_1, b_2\})$ (resp. to (G, B)) a copy of an auxiliary graph $BS(b_1, b_2)$, then the following assertions about the resulting graph G' are true.*

- *The resulting graph G' is weakly chordal (resp. (3, 1)-polar).*
- *If all cliques of G have size at least 3, then all cliques of G' have size at least 3.*
- *Any 2-clique-colouring of G' uses 2 colours in the set of vertices $\{b_1, b_2\}$.*
- *If G' is 2-clique-colourable, then G is 2-clique-colourable.*
- *If there exists a 2-clique-colouring of G that uses 2 colours in the set of vertices $\{b_1, b_2\}$, then G' is 2-clique-colourable.*

Proof. Let $G = (V, E)$ be a weakly chordal graph and let adjacent vertices $b_1, b_2 \in V$. Add to $(G, \{b_1, b_2\})$ a copy of an auxiliary graph $BS(b_1, b_2)$ in order to obtain graph G'. First, we prove that $BS(b_1, b_2)$ is a weakly chordal (3, 1)-polar graph. A complete graph K_5 with vertices b_1, b_2, b', b'', b''' is a weakly chordal (3, 1)-polar graph with vertex partition (A, B), where $A = \emptyset$ and $B = \{b_1, b_2, b', b'', b'''\}$. By Lemma 7, if we add copies of the auxiliary graphs $BP(b_1, b_2, b')$, $BP(b_1, b_2, b'')$, $BP(b_1, b_2, b''')$, and $BP(b', b'', b''')$, then we have a weakly chordal (3, 1)-polar graph that corresponds to $BS(b_1, b_2)$. Hence, $BS(b_1, b_2)$ is a weakly chordal (3, 1)-polar graph.

Suppose, by contradiction, that G' is not weakly chordal, which means that G' has W a chordless cycle with a number of vertices greater than 4 or the complement of a chordless cycle with a number of vertices greater than 5. Since both G and $BS(b_1, b_2)$ are weakly chordal graphs, a chordless cycle W with a number of vertices greater than 4 contains a vertex of $BS(b_1, b_2) \setminus G$ and a vertex of $G \setminus BS(b_1, b_2)$. Note that $\{b_1, b_2\}$ induces a complete subgraph and is a cutset of G' that disconnects $BS(b_1, b_2) \setminus G$ from $G \setminus BS(b_1, b_2)$. So, every cycle with vertices of $BS(b_1, b_2) \setminus G$ and of $G \setminus BS(b_1, b_2)$ contains a chord, i.e. there is no such chordless cycle W. For W the complement a chordless cycle with a number of vertices greater than 5, W has vertices b_1 or b_2, otherwise W is disconnected. Hence, W has a 1-cutset or a 2-cutset, which is a contradiction since the complement of a chordless cycle with a number of vertices greater than 5 is triconnected. Hence, G' is weakly chordal. If $G = (V, E)$ is a (3, 1)-polar graph with partition (A, B) and $b_1, b_2 \in B$, then G' is obtained by adding to (G, B) a copy of $BS(b_1, b_2)$, and G' is a (3, 1)-polar graph with partition (A', B'), where $A' = A \cup (V(BS(b_1, b_2)) \setminus \{b_1, b_2, b', b'', b'''\})$ and $B' = B \cup \{b', b'', b'''\}$ as the partition of $V(G')$ into two sets. Notice that all added satellites are triangles. Hence, G' is a (3, 1)-polar graph.

The cliques of G' that are not cliques of G are precisely the clique containing $\{b_1, b_2, b', b'', b'''\}$, and all cliques added by the inclusion of copies of the auxiliary graphs $BP(b_1, b_2, b')$, $BP(b_1, b_2, b'')$, $BP(b_1, b_2, b''')$, and $BP(b', b'', b''')$. By Lemma 7, all cliques added by the auxiliary graphs have size at least 3. Then, all cliques of G' have size at least 3.

Consider any 2-clique-colouring π' of G'. Since we added a copy of the auxiliary graph $BP(b', b'', b''')$, Lemma 7 states that π' assigns at least 2 colours to b', b'', b'''. Without loss of generality, suppose that π' assigns distinct colours to b' and b''. Since we added copies of the auxiliary graphs $BP(b', b_1, b_2)$ and $BP(b'', b_1, b_2)$, Lemma 7 states that π' assigns at least 2 colours to $\{b', b_1, b_2\}$ and at least 2 colours to $\{b'', b_1, b_2\}$, i.e. π' assigns a colour which is not $\pi'(b')$ to b_1 or b_2 and a colour which is not $\pi'(b'')$ to b_1 or b_2. Hence, π' assigns 2 distinct colours to b_1, b_2.

Since every 2-clique-colouring π' of G' uses 2 colours in the set $\{b_1, b_2\}$ and every clique of G that is not a clique of G' contains $\{b_1, b_2\}$, π' defines a 2-clique-colouring π of G.

Now, consider a 2-clique-colouring π of G that assigns 2 colours to b_1, b_2. Assign the same colour of b_1 to b' and b''. Assign the same colour of b_2 to b'''. The sets $\{b_1, b_2, b'\}$, $\{b_1, b_2, b''\}$, $\{b_1, b_2, b'''\}$, and $\{b', b'', b'''\}$ have 2 colours each. By Lemma 7, all cliques added by the inclusion of copies of the auxiliary graphs $BP(b_1, b_2, b')$, $BP(b_1, b_2, b'')$, $BP(b_1, b_2, b''')$, and $BP(b', b'', b''')$ are polychromatic. Hence, we have a 2-clique-colouring π' of G'. $\square$

Recall that $(3, 1)$-polar graphs with all cliques having size at least 3 are weakly chordal, since $\overline{C_6}$ has cliques of size 2 and $\overline{C_{2t}}$ for $t > 3$ is not $(3, 1)$-polar. We strengthen the result that 2-clique-colouring of $(3, 1)$-polar graphs is $\mathcal{NP}$-complete, now even restricting all cliques to have size at least 3, which gives a subclass of weakly chordal graphs.

Theorem 9. *The problem of 2-clique-colouring is $\mathcal{NP}$-complete for (3, 1)-polar graphs with all cliques having size at least 3.*

Proof. The problem of 2-clique-colouring a (3, 1)-polar graph with all cliques having size at least 3 is in $\mathcal{NP}$: Theorem 5 confirms that to check whether a 2-colouring of a (3, 1)-polar graph is a 2-clique-colouring is in $\mathcal{P}$.

We prove that 2-clique-colouring (3, 1)-polar graphs with all cliques having size at least 3 is $\mathcal{NP}$-hard by reducing NAE-3SAT [12] to it. We apply the ideas of the framework developed by Kratochvíl and Tuza [7]. For every formula ϕ, a (3, 1)-polar graph G with all cliques having size at least 3 is constructed such that ϕ is satisfiable if, and only if, graph G is 2-clique-colourable. Let n (resp. m) be the number of variables (resp. clauses) in formula ϕ. We define graph G as follows.

- for each variable x_i, $1 \le i \le n$, we create two vertices x_i and $\overline{x}_i$. Moreover, we create edges so that the set $\{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$ induces a complete subgraph of G.
- for each variable x_i, $1 \le i \le n$, add a copy of the auxiliary graph $BS(x_i, \overline{x}_i)$. Vertices x_i and $\overline{x}_i$ correspond to the literals of variable x_i.
- for each clause $c_j = (l_a, l_b, l_c)$, $1 \le j \le m$, we add a copy of the auxiliary graph $BP(l_a, l_b, l_c)$.

Refer to Fig. 6 for an example of such construction, given a formula $\phi = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee x_5) \wedge (x_1 \vee x_3 \vee x_5)$.

First, we prove that the graph G is a (3, 1)-polar graph with all cliques having size at least 3.

Consider the set $\{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$. Clearly, this set is a clique with size at least 3 and also a (3, 1)-polar graph. Lemma 8 states that, for each added auxiliary graph $BS(x_i, \overline{x}_i)$ to a (3, 1)-polar graph with all cliques having size at least 3, every obtained graph remains in the class. Lemma 7 states that, for each added auxiliary graph $BP(l_{a_{c_j}}, l_{b_{c_j}}, l_{c_{c_j}})$ to a (3, 1)-polar graph with all cliques having size at least 3, every obtained graph remains in the class. Hence, G is a (3, 1)-polar graph with all cliques having size at least 3.

Such construction is done in polynomial-time. One can check with Lemmas 7 and 8 that G has $3m + 17n$ vertices.

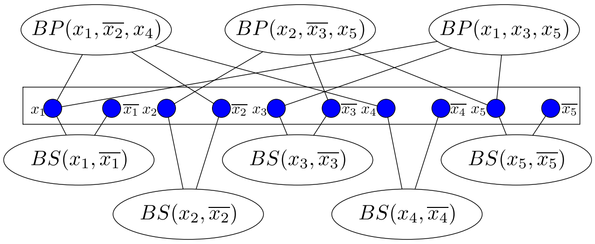


Fig. 6. Example of a (3, 1)-polar graph with all cliques having size at least 3 constructed for a NAE-3SAT instance $\phi = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee x_5) \wedge (x_1 \vee x_3 \vee x_5)$.

We claim that formula ϕ is satisfiable if, and only if, there exists a 2-clique-colouring of G. Assume there exists a valuation v_ϕ such that ϕ is satisfied. We give a colouring to graph G, as follows.

- assign colour 1 to $l \in \{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$ if it corresponds to the literal which receives the *true* value in v_ϕ, otherwise we assign colour 2 to it.
- extend the 2-clique-colouring to the copy of the auxiliary graph $BS(x_i, \overline{x}_i)$, for each variable x_i, $1 \le i \le n$, according to Lemma 8. Notice that the necessary condition to extend the 2-clique-colouring is satisfied.
- extend the 2-clique-colouring to the copy of the auxiliary graph $BP(l_a, l_b, l_c)$, for each triangle $c = \{l_a, l_b, l_c\}$, $1 \le j \le m$, according to Lemma 7. Notice that the necessary condition to extend the 2-clique-colouring is satisfied.

It still remains to be proved that this is indeed a 2-clique-colouring.

Consider the set $\{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$. Clearly, the above colouring assigns 2 colours to this set. Lemma 8 states that, for each added auxiliary graph $BS(x_i, \overline{x}_i)$ to a 2-clique-colourable (3, 1)-polar graph, we obtain a 2-clique-colourable graph. Lemma 7 states that, for each added auxiliary graph $BP(l_a, l_b, l_c)$ to a 2-clique-colourable (3, 1)-polar graph, we obtain a 2-clique-colourable graph. Hence, graph G is 2-clique-colourable.

For the converse, we now assume that G is 2-clique-colourable and we consider any 2-clique-colouring. Recall that, by Lemma 8, the vertices x_i and $\overline{x}_i$ have distinct colours, since we added the auxiliary graph $BS(x_i, \overline{x}_i)$, for each variable x_i. Hence, we define v_ϕ as follows. The literal x_i is assigned *true* in v_ϕ if the corresponding vertex has colour 1 in the clique-colouring, otherwise it is assigned *false*. Since we are considering a 2-clique-colouring, by Lemma 7, every triangle c_j, $1 \le j \le m$, is polychromatic. As a consequence, there exists at least one literal with *true* value and at least one literal with *false* value in every clause c_j. This proves that ϕ is satisfied for valuation v_ϕ. □

On the other hand, we prove that 2-clique-colouring (2, 1)-polar graphs becomes polynomial when all cliques have size at least 3.

Theorem 10. *Every (2, 1)-polar graph with all cliques having size at least 3 is 2-clique-colourable, and such a 2-clique-colouring can be determined in polynomial time.*

Proof. Let $G = (V, E)$ be a (2, 1)-polar graph with all cliques having size at least 3, and with vertex partition (A, B). We first colour the vertices of B: if B is a singleton, then assign colour 1 to its only vertex; if $|B| > 1$, then colour the vertices of B using precisely two colours 1 and 2, arbitrarily.

Next, we colour the vertices of A. For each singleton satellite $S = \{s\}$, assign colour 1 to vertex s, if s has a neighbour in B with colour 2, otherwise assign to vertex s colour 2.

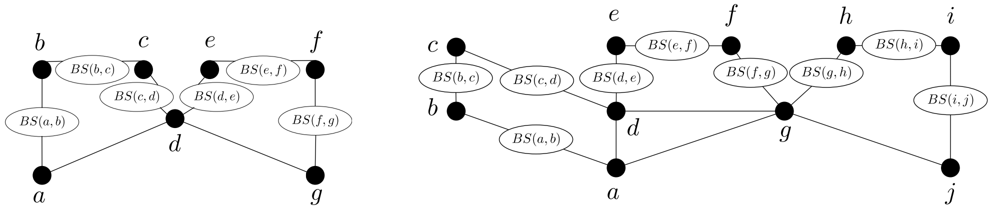
The other case considers each edge satellite $S = \{s_1, s_2\}$, where we note that set S is not a clique, which implies that $N_B(s_1) \cap N_B(s_2) \neq \emptyset$. There are just three cliques containing set S: the set $S \cup (N_B(s_1) \cap N_B(s_2))$, the set $\{s_1\} \cup N_B(s_1)$, and the set $\{s_2\} \cup N_B(s_2)$. Note that every clique that contains set S also contains set $N_B(s_1) \cap N_B(s_2)$. Assign colour 1 to both vertices s_1 and s_2, if there is a vertex in $N_B(s_1) \cap N_B(s_2)$ with colour 2, otherwise assign to both vertices s_1 and s_2 colour 2.

Clearly, such assignment is a 2-clique-colouring and can be achieved in polynomial time. □

In the proof that 2-clique-colouring weakly chordal graphs is a Σ_2^P-complete problem (Theorem 3), we constructed a weakly chordal graph with K_2 cliques to force distinct colours in their extremities (in a 2-clique-colouring). We can eliminate the K_2 cliques and obtain a weakly chordal graph with all cliques having size at least 3 by adding copies of the auxiliary graph $BS(u, v)$, for every K_2 clique $\{u, v\}$. Auxiliary graphs AK and NAS in Fig. 2 become AK' and NAS', both depicted in Fig. 7.

With this modification, the weakly chordal graph constructed in Theorem 3 becomes a weakly chordal graph with no K_2 clique, depicted in Fig. 8.

Such construction is done in polynomial-time. Notice that, in the constructed graph of Theorem 3, every K_2 clique $\{u, v\}$ has 2 distinct colours in a clique-colouring. Hence, one can check with Lemmas 7 and 8 that the obtained graph is weakly



(a) $AK'(a,g)$ (b) $NAS'(a,j)$

Fig. 7. Auxiliary graphs $AK'(a,g)$ and $NAS'(a,j)$.

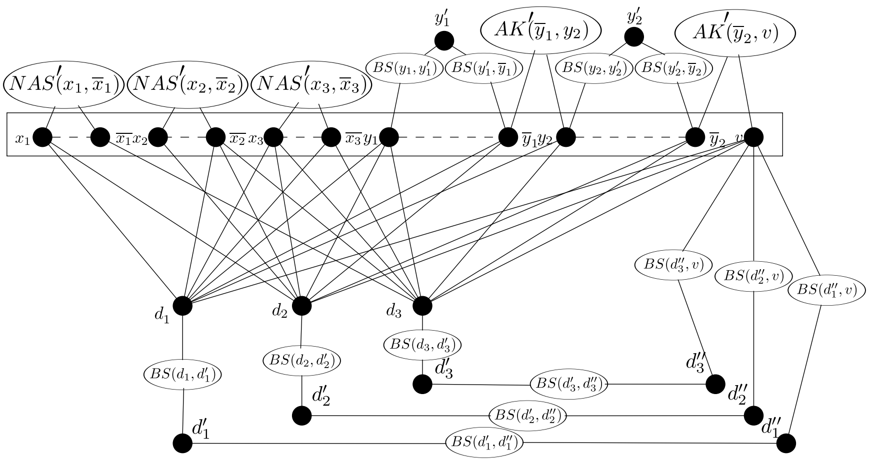


Fig. 8. Graph constructed for a QSAT2 instance $\Psi = (x_1 \wedge \overline{x}_2 \wedge y_2) \vee (x_1 \wedge x_3 \wedge \overline{y}_2) \vee (\overline{x}_1 \wedge \overline{x}_2 \wedge y_1)$.

chordal and it is 2-clique-colourable if, and only if, the constructed graph of Theorem 3 is 2-clique-colourable. This implies the following theorem.

Theorem 11. *The problem of 2-clique-colouring is Σ_2^P-complete for weakly chordal graphs with all cliques having size at least 3.*

As a direct consequence of Theorem 11, we have that 2-clique-colouring is Σ_2^P-complete for perfect graphs with all cliques having size at least 3.

Corollary 12. *The problem of 2-clique-colouring is Σ_2^P-complete for perfect graphs with all cliques having size at least 3.*

4. Final considerations

Marx [9] proved complexity results for k-clique-colouring, for fixed $k \geq 2$, and related problems that lie in between two distinct complexity classes, namely Σ_2^P-complete and Π_3^P-complete. Marx approaches the complexity of clique-colouring by fixing the graph class and diversifying the problem. In the present work, our point of view is the opposite: we rather fix the (2-clique-colouring) problem and we classify the problem complexity according to the input graph class, which belongs to nested subclasses of weakly chordal graphs. We achieved complexities lying in between three distinct complexity classes, namely Σ_2^P-complete, $\mathcal{NP}$-complete and $\mathcal{P}$. Fig. 9 shows the relation of inclusion among the classes of graphs of Table 1. The 2-clique-colouring complexity for each class is highlighted.

Notice that the perfect graph subclasses for which the 2-clique-colouring problem is in $\mathcal{NP}$ mentioned so far in the present work satisfy that the number of cliques is polynomial. We remark that the complement of a matching has an exponential number of cliques and yet the 2-clique-colouring problem is in $\mathcal{NP}$, since every such graph is 2-clique-colourable. Now, notice that the perfect graph subclasses for which the 2-clique-colouring problem is in $\mathcal{P}$ mentioned so far in the present work satisfy that all graphs in the class are 2-clique-colourable. Macêdo Filho et al. [6] have proved that unichord-free graphs are 3-clique-colourable, but a unichord-free graph is 2-clique-colourable if and only if it is perfect. As a future work, we aim to find interesting subclasses of perfect graphs where not all graphs are 2-clique-colourable and yet the 2-clique-colouring problem is in $\mathcal{P}$ when restricted to the class.

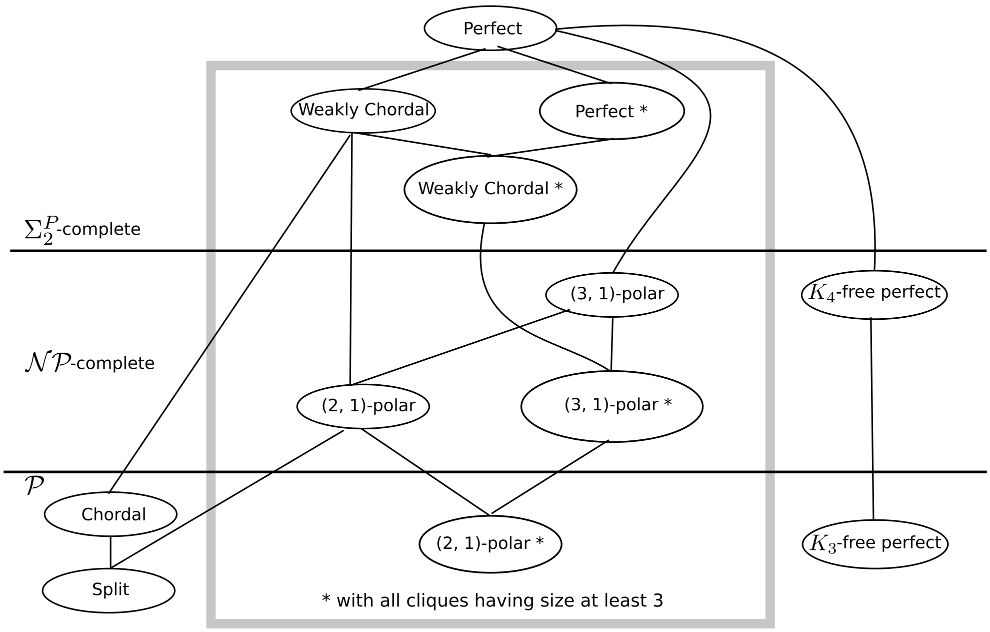


Fig. 9. 2-clique-colouring complexity of perfect graphs and subclasses. Inside the frame, we have the graph classes considered in the present paper.

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