Chapter 2: Simplification, Optimization and Implication

Where we learn more fun things we can do with constraints.

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Simplification, Optimization and Implication

- Constraint Simplification
- Projection
- Constraint Simplifiers
- Optimization
- Implication and Equivalence

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Constraint Simplification

- Two equivalent constraints represent the same information, **but**
- One may be simpler than the other

$$
\begin{aligned}
& X \geq 1 \wedge X \geq 3 \wedge 2 = Y + X \\
& \leftrightarrow X \geq 3 \wedge 2 = Y + X \\
& \leftrightarrow 3 \leq X \wedge X = 2 - Y \\
& \leftrightarrow X = 2 - Y \wedge 3 \leq X \\
& \leftrightarrow X = 2 - Y \wedge 3 \leq 2 - Y \\
& \leftrightarrow X = 2 - Y \wedge Y \leq -1
\end{aligned}
$$

Removing redundant constraints, rewriting a primitive constraint, changing order, substituting using an equation all preserve equivalence

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Redundant Constraints

- One constraint *C1* **implies** another *C2* if the solutions of *C1* are a subset of those of *C2*
- *C2* is said to be **redundant** wrt *C1*
- It is written $C1 \rightarrow C2$

$$X \geq 3 \rightarrow X \geq 1$$

$$Y \leq X + 2 \wedge Y \geq 4 \rightarrow X \geq 1$$

$$cons(X, X) = cons(Z, nil) \rightarrow Z = nil$$

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Redundant Constraints

- We can remove a primitive constraint which is redundant with respect to the rest of the constraint.

$$\underline{X \geq 1} \wedge X \geq 3 \leftrightarrow X \geq 3$$

$$Y \leq X + 2 \wedge \underline{X \geq 1} \wedge Y \geq 4 \leftrightarrow Y \leq X + 2 \wedge Y \geq 4$$

$$cons(X, X) = cons(Z, nil) \wedge \underline{Z = nil} \leftrightarrow cons(X, X) = cons(Z, nil)$$

Definitely produces a simpler constraint

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Solved Form Solvers

- Since a solved form solver creates equivalent constraints it can be a simplifier

For example using the term constraint solver

$$cons(X, X) = cons(Z, nil) \wedge Y = succ(X) \wedge succ(Z) = Y \wedge Z = nil$$

$$\leftrightarrow X = nil \wedge Z = nil \wedge Y = succ(nil)$$

Or using the Gauss-Jordan solver

$$X = 2 + Y \wedge 2Y + X - T = Z \wedge X + Y = 4 \wedge Z + T = 5$$

$$\leftrightarrow X = 3 \wedge Y = 1 \wedge Z = 5 - T$$

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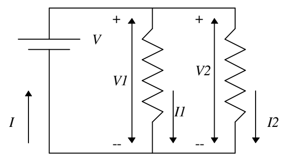
Projection

It becomes even more important to simplify when we are only interested in some variables in the constraint

$$
\begin{aligned}
V1 &= I1 \times R1 \\
V2 &= I2 \times R2 \\
V - V1 &= 0 \\
V - V2 &= 0 \\
V1 - V2 &= 0 \\
I - I1 - I2 &= 0 \\
-I + I1 + I2 &= 0 \\
R1 &= 5 \\
R2 &= 10
\end{aligned}
$$



Simplified wrt to V and I $\quad V = \frac{10}{3} I$

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Projection

- The **projection** of a constraint C onto variables V is a constraint $C1$ such that
  - $C1$ only involves variables V
  - Every solution of C is a solution of $C1$
  - A solution of $C1$ can be extended to give a solution of C

$X \geq Y \wedge Y \geq Z \wedge Z \geq 0 \qquad X \geq 0$

$\{X \mapsto 0, Y \mapsto 0, Z \mapsto 0\} \qquad \{X \mapsto 0\}$

$\{X \mapsto 4, Y \mapsto 3, Z \mapsto 1\} \qquad \{X \mapsto 4\}$

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Fouriers Algorithm

- Eliminates variable y from linear ineqs C
- Write each ineq with y on one side of ineq
- $t_1 \leq y \qquad y \leq t_2$
- For each pair $\quad t_1 \leq y \quad y \leq t_2$
  - produce a new ineq $\; t_1 \leq y \otimes y \leq t_2 \qquad t_1 \leq t_2$
- The result is all new ineqs and those ineqs in C which do not involve y

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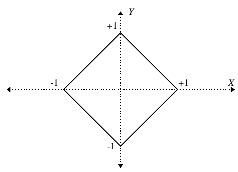
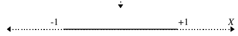
Fouriers Algorithm Example

Projecting out Y

$$X-1 \le Y$$
$$-1-X \le Y$$
$$Y \le 1-X$$
$$Y \le 1+X$$



$$X-1 \le Y \otimes Y \le 1-X \quad X \le 1$$
$$X-1 \le Y \otimes Y \le 1+X \quad 0 \le 2$$
$$-1-X \le Y \otimes Y \le 1-X \quad 0 \le 2$$
$$-1-X \le Y \otimes Y \le 1+X \quad -1 \le X$$



Result only involving X

$$X \le 1 \wedge -1 \le X$$

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Projecting Term Constraints

- We can also project term constraints
- $cons(Y,Y) = cons(X,Z) \wedge succ(Z) = succ(T)$
- projected onto *{X,Z}* is $X = Z$
- But what is $X = cons(Y,Z)$ projected onto X?
- Answer: there is no such constraint!

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Constraint Simplifiers

- constraints *C1* and *C2* are **equivalent wrt variables** *V* if
  - taking any solution of one and restricting it to the variables *V*, this restricted solution can be extended to be a solution of the other
- Example X=*succ(Y)* and X=*succ(Z)* wrt *{X}*

$X = succ(Y)$	$\{X\}$	$X = succ(Z)$
$\{X \mapsto succ(a), Y \mapsto a\}$	$\{X \mapsto succ(a)\}$	$\{X \mapsto succ(a), Z \mapsto a\}$
$\{X \mapsto succ(nil), Y \mapsto nil\}$	$\{X \mapsto succ(nil)\}$	$\{X \mapsto succ(nil), Z \mapsto nil\}$

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Simplifier Definition

- A **constraint simplifier** is a function *simpl* which takes a constraint *C* and set of variables *V* and returns a constraint *C1* that is equivalent to *C* wrt *V*
- We can make a simplifier for real inequalities from Fouriers algorithm

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Tree Constraint Simplifier

- apply the term solver to *C* obtaining *C1*
- if *C1* is *false* then return *false*
- foreach equation *x=t* in *C1*
  - if *x* is in *V* then
    - if *t* is a variable not in *V*
      - substitute *x* for *t* throughout *C1* and result
    - else add *x=t* to result
- return result

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Tree Simplification Example

Tree constraint to be simplified wrt *{Y,T}*

$$h(f(X,Y),,g(T)) = h(f(g(T),X), f(X,X), g(U))$$

Equivalent constraint from tree solver

$$Z = f(g(U), g(U)) \wedge X = g(U) \wedge Y = g(U) \wedge T = U$$

Discard the first two equations, keep the third and use the last to substitute for *U* by *T*

$$Y = g(T)$$

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Simplifier Properties

- Desirable properties of a simplifier are:
  - **projecting**: $vars(simpl(C,V)) \subseteq V$
  - **weakly projecting**: for all constraints *C2* that are equivalant to *C1* wrt *V*
    - $|vars(simpl(C1,V)) - V| \leq |vars(C2) - V|$
    - a weakly projecting solver never uses more variables than is required
- Both properties allow a simplifier to be used as a solver (How?)

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Optimization

- Often given some problem which is modelled by constraints we dont want just any solution, but a "best" solution
- This is an **optimization problem**
- We need an **objective function** so that we can rank solutions, that is a mapping from solutions to a real value

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Optimization Problem

- An **optimization problem** *(C,f)* consists of a constraint *C* and objective function *f*
- A valuation *v1* is **preferred** to valuation *v2* if *f(v1) < f(v2)*
- An **optimal solution** is a solution of *C* such that no other solution of *C* is preferred to it.

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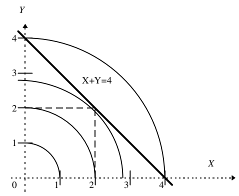
Optimization Example

An optimization problem

$(C \equiv X + Y \geq 4, \quad f \equiv X^2 + Y^2)$

Find the closest point to the origin satisfying the *C*.
Some solutions and *f* value

$\{X \mapsto 0, Y \mapsto 4\}$	16
$\{X \mapsto 3, Y \mapsto 3\}$	18
$\{X \mapsto 2, Y \mapsto 2\}$	8



Optimal solution
$\{X \mapsto 2, Y \mapsto 2\}$

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Optimization

- Some optimization problems have no solution.
  - Constraint has no solution
  - $(X \geq 2 \wedge X \leq 0, \ X^2)$
  - Problem has no optimum
  - $(X \leq 0, \ X)$
  - For any solution there is more preferable one

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Simplex Algorithm

- The most widely used optimization algorithm
- Optimizes a linear function wrt to linear constraints
- Related to Gauss-Jordan elimination

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Simplex Algorithm

- A optimization problem (C, f) is in **simplex form**:
  - C is the conjunction of CE and CI
  - CE is a conjunction of linear equations
  - CI constrains all variables in C to be non-negative
  - f is a linear expression over variables in C

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Simplex Example

An optimization problem in simplex form

minimize $3X+2Y-Z+1$ subject to

$$\begin{array}{lllll} X & +Y & & & = 3 \land \\ -X & -3Y & +2Z & +T & = 1 \land \\ \multicolumn{5}{l}{X \geq 0 \land Y \geq 0 \land Z \geq 0 \land T \geq 0} \end{array}$$

- An arbitrary problem can be put in simplex form by
  - replacing unconstrained var X by new vars $X^+ - X^-$
  - replacing ineq $e \leq r$ by new var s and $e + s = r$

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Simplex Solved Form

- A simplex optimization problem is in **basic feasible solved (bfs) form** if:
  - The equations are in solved form
  - Each constant on the right hand side is non-negative
  - Only parameters occur in the objective
- A **basic feasible solution** is obtained by setting each parameter to 0 and each non-parameter to the constant in its equation

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Simplex Example

An equivalent problem to that before in bfs form

minimize $10 - Y - Z$ subject to

$$\begin{array}{lllll} X = & 3 & -Y & & \wedge \\ T = & 4 & +2Y & -2Z & \wedge \end{array}$$
$$X \ge 0 \wedge Y \ge 0 \wedge Z \ge 0 \wedge T \ge 0$$

We can read off a solution and its objective value

$$\{X \mapsto 3, T \mapsto 4, Y \mapsto 0, Z \mapsto 0\}$$
$$f = 10$$

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Simplex Algorithm

starting from a problem in bfs form

repeat

Choose a variable y with negative coefficient in the obj. func.

Find the equation $x = b + cy + \ldots$ where $c<0$ and $-b/c$ is minimal

Rewrite this equation with y the subject $y = -b/c + 1/c\ x + \ldots$

Substitute $-b/c + 1/c\ x + \ldots$ for y in all other eqns and obj. func.

until no such variable y exists or no such equation exists

if no such y exists optimum is found

else there is no optimum solution

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Simplex Example

minimize $10 - Y - Z$ subject to

$$\begin{array}{lllll} X = & 3 & -Y & & \wedge \\ T = & 4 & +2Y & -2Z & \wedge \end{array}$$
$$X \ge 0 \wedge Y \ge 0 \wedge Z \ge 0 \wedge T \ge 0$$

Choose variable *Y*, the first eqn is only one with neg. coeff $Y = 3 - X$

minimize $7 + X - Z$ subject to

$$\begin{array}{lllll} Y = & 3 & -X & & \wedge \\ T = & 10 & -2X & -2Z & \wedge \end{array}$$
$$X \ge 0 \wedge Y \ge 0 \wedge Z \ge 0 \wedge T \ge 0$$

Choose variable *Z*, the 2nd eqn is only one with neg. coeff $Z = 5 - X - 0.5T$

minimize $2 + 2X + 0.5T$ subject to

$$\begin{array}{lllll} Y = & 3 & -X & & \wedge \\ Z = & 5 & -X & -0.5T & \wedge \end{array}$$
$$X \ge 0 \wedge Y \ge 0 \wedge Z \ge 0 \wedge T \ge 0$$

No variable can be chosen, optimal value 2 is found

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Another example

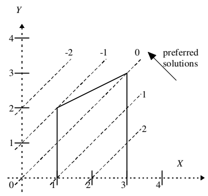
minimize $X - Y$ subject to

$$
\begin{aligned}
&Y \geq 0 \wedge \\
&X \geq 1 \wedge \\
&X \leq 3 \wedge \\
&2Y \leq X + 3
\end{aligned}
$$

An equivalent simplex form is:

$$
\begin{array}{llllll}
X & & & - S_2 & & = 1 \wedge \\
X & & & & + S_3 & = 3 \wedge \\
-X & + 2Y & - S_1 & & & = 3 \wedge
\end{array}
$$

$$X \geq 0 \wedge Y \geq 0 \wedge S_1 \geq 0 \wedge S_2 \geq 0 \wedge S_3 \geq 0$$



An optimization problem showing contours of the objective function

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Another example

Basic feasible solution form: circle

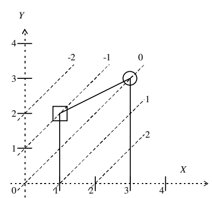
minimize $0 + 0.5S_1 - 0.5S_3$ subject to

$$
\begin{array}{llll}
Y = & 3 & -0.5S_1 & -0.5S_3 \quad \wedge \\
S_2 = & 2 & & -S_3 \quad \wedge \\
X = & 3 & & -S_3 \quad \wedge
\end{array}
$$

Choose S3, replace using 2nd eq

minimize $-1 + 0.5S_1 + 0.5S_2$ subject to

$$
\begin{array}{llll}
Y = & 3 & -0.5S_1 & +0.5S_2 \quad \wedge \\
S_3 = & 2 & & -S_2 \quad \wedge \\
X = & 1 & & +S_2 \quad \wedge
\end{array}
$$



Optimal solution: box

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The Missing Part

- How do we get the initial basic feasible solution?
- Solve a different simplex problem
  - Add artificial variables to make equations in basic feasible solved form
  - Minimize the sumof the artificial variables
  - If the sum is zero we can construct a bfs for the original problem

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The Missing Part Example

Original simplex form equations

$$\begin{array}{llll} X & & -S_2 & = 1 \wedge \\ X & & +S_3 & = 3 \wedge \\ -X & +2Y & -S_1 & = 3 \end{array}$$

With artificial vars in bfs form:

$$\begin{array}{l} A_1 = 1 - X + S_2 \wedge \\ A_2 = 3 - X - S_3 \wedge \\ A_3 = 3 + X - 2Y - S_1 \end{array}$$

Objective function: minimize

$$A_1 + A_2 + A_3$$
$$= 7 - X - 2Y - S_1 + S_2 - S_3$$

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Missing Part Example II

Problem after minization of objective function

minimize $A_1 + A_2 + A_3$ subject to

$$\begin{array}{lllllll} Y & = & 3 & -0.5S_1 & -0.5S_3 & -0.5A_2 & -0.5A_3 \wedge \\ S_2 & = & 2 & & -S_3 & +A_1 & -A_2 \wedge \\ X & = & 3 & & -S_3 & -A_2 & \end{array}$$

Removing the artificial variables, the original problem

$$\begin{array}{lllll} Y = & 3 & -0.5S_1 & -0.5S_3 & \wedge \\ S_2 = & 2 & & -S_3 & \wedge \\ X = & 3 & & -S_3 & \wedge \end{array}$$

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Implication and Equivalence

- Other important operations involving constraints are:
- **implication**: test if *C1* implies *C2*
  - *impl(C1, C2)* answers *true, false* or *unknown*
- **equivalence**: test if *C1* and *C2* are equivalent
  - *equiv(C1, C2)* answers *true, false* or *unknown*

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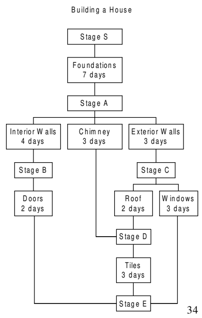
Implication Example

For the house constraints *CH*, will stage B have to be reached after stage C?

$$CH \rightarrow T_B \geq T_C$$

For this question the answer if *false*, but if we require the house to be finished in 15 days the answer is *true*

$$CH \wedge T_E = 15 \rightarrow T_B \geq T_C$$



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Implication and Equivalence

- We can use *impl* to define *equiv* and vice versa
- $impl(C1, C2) = equiv(C1, C1 \wedge C2)$
- $equiv(C1, C2) = impl(C1, C2) \wedge impl(C2, C1)$
- We can use a solver to do simple *impl* tests
- $impl(C1, C2) = \neg solv(C1 \wedge \neg C2)$
- e.g. $impl(CH, T_B \geq T_C) = \neg solv(CH \wedge T_B < T_C)$

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Simplication, Optimization and Implication Summary

- Equivalent constraints can be written in many forms, hence we desire simplification
- Particularly if we are only interested in the interaction of some of the variables
- Many problems desire a optimal solution, there are algms (simplex) to find them
- We may also be interested in asking questions involving implication

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